

# QUANTUM WALLED BRAUER ALGEBRA: COMMUTING FAMILIES, BAXTERIZATION, AND REPRESENTATIONS

A. M. SEMIKHATOV AND I. YU. TIPUNIN

**ABSTRACT.** For the quantum walled Brauer algebra, we construct its Specht modules and (for generic parameters of the algebra) seminormal modules. The latter construction yields the spectrum of a commuting family of Jucys–Murphy elements. We also propose a Baxterization prescription; it involves representing the quantum walled Brauer algebra in terms of morphisms in a braided monoidal category and introducing parameters into these morphisms, which allows constructing a “universal transfer matrix” that generates commuting elements of the algebra.

## 1. INTRODUCTION

We study the quantum (“quantized”) walled Brauer algebra  $\text{qw}\mathcal{B}$  and its representations.<sup>1</sup> The classical version of the algebra was introduced in [1, 2] in the context of generalized Schur–Weyl duality: the algebra was shown to centralize the  $gl(N)$  action on “mixed” tensor products  $X^{*\otimes m} \otimes X^{\otimes n}$  of the natural  $gl(N)$  representation and its dual; for special parameter values, the walled Brauer algebra centralizes the action of  $gl(M|N)$  [3, 4]. The structure of the algebra was explored in [5]. The quantum version of the algebra was introduced in [6, 7, 8, 9], and its role as the centralizer of  $U_q(gl_N)$  on the mixed tensor product  $X^{*\otimes m} \otimes X^{\otimes n}$  was elucidated in [10, 11]. We also note a recent “super” extension of quantum walled Brauer algebras in [12].

We here view  $\text{qw}\mathcal{B}_{m,n}$  following [7] (also see [13]) as a diagram algebra, with the diagrams supplied by a braided monoidal category.<sup>2</sup> We use the diagrams representing category morphisms to construct two types of commutative families of  $\text{qw}\mathcal{B}_{m,n}$  elements:

- (i) a family of “conservation laws” following from a Baxterization procedure, and
- (ii) a family of Jucys–Murphy elements  $J(n)_2, \dots, J(n)_{m+n} \in \text{qw}\mathcal{B}_{m,n}$  (which we diagonalize, as is discussed below).

The commuting  $\text{qw}\mathcal{B}_{m,n}$  elements in item (i) are called “conservation laws” or “Hamiltonians” in view of the physical interpretation of (quantum) walled Brauer algebras as per-

<sup>1</sup>Speaking of an algebra rather than algebras, we mean a particular member of a family  $\text{qw}\mathcal{B}_{m,n}$  of quantum walled Brauer algebras with  $m, n \geq 1$ ; this algebra, moreover, depends on parameters, as we discuss below.

<sup>2</sup>To be developed in full rigor, this approach would require a braided version of Deligne’s category [14], which can apparently be done, but is beyond the scope of this paper (see [15] and the references therein).

taining to integrable models of statistical mechanics, e.g., the  $t$ – $J$  model (see [16, 17, 18] and the references therein). The commuting elements follow by expanding a “universal transfer matrix”  $\mathcal{A}_{m,n}(z)$  in the spectral parameter  $z \in \mathbb{C}$  introduced by a trick that generalizes the well-known Baxterization of Hecke algebras. For  $\text{qw}\mathcal{B}_{m,n}$ , however, it is applied not only to the algebra generators but also to morphisms in a braided monoidal category  $\mathbb{B}_{\bullet\circ}$  that do not belong to  $\text{qw}\mathcal{B}_{m,n}$ . Using these, it is straightforward to show that  $[\mathcal{A}_{m,n}(z), \mathcal{A}_{m,n}(w)] = 0$ , but at the same time it turns out that  $\mathcal{A}_{m,n}(z) \in \text{qw}\mathcal{B}_{m,n} \otimes \mathbb{C}(z)$ , whence a commuting family of  $\text{qw}\mathcal{B}_{m,n}$  elements follows by expanding around a suitable value of the spectral parameter.

By borrowing more from integrable systems of statistical mechanics, but staying within the  $\text{qw}\mathcal{B}_{m,n}$  algebra, it would be quite interesting to diagonalize our conservation laws by Bethe-ansatz techniques, but we here solve only a more modest diagonalization problem, the one for a family of Jucys–Murphy elements (which are in fact the  $z \rightarrow \infty$  limits of “monodromy matrices” constructed similarly to the transfer matrices).

We recall that Jucys–Murphy elements were originally introduced for the symmetric group algebra [19, 20, 21] and were then discussed for some other diagram algebras [22, 23, 24] and in even broader contexts (see, e.g., [25, 26, 13, 27]), up to the generality of cellular algebras [28, 29]. As with the conservation laws, we here define Jucys–Murphy elements  $J(n)_j \in \text{qw}\mathcal{B}_{m,n}$ ,  $j = 2, \dots, m+n$ , in a way that makes their commutativity manifest, but involves braided-category morphisms not from  $\text{qw}\mathcal{B}_{m,n}$ ; again, their apparent ill-definedness is in fact superficial, and the  $J(n)_j$  can eventually be expressed via relatively explicit formulas in terms of generators.

For generic values of the algebra parameters, when  $\text{qw}\mathcal{B}_{m,n}$  is semisimple, we diagonalize the commuting family of Jucys–Murphy elements in each irreducible representation. This amounts to constructing seminormal  $\text{qw}\mathcal{B}_{m,n}$  representations. Seminormal bases/representations have been studied rather extensively for a number of “related” algebras (see, e.g., [30, 26, 27, 31, 28, 32]), but apparently not for  $\text{qw}\mathcal{B}_{m,n}$ . The seminormal basis is made of triples of Young tableaux of certain shapes, and an essential novelty compared with the Hecke-algebra case is the “mobile elements”—corners of Young tableaux that can change their position.

For special parameter values,  $\text{qw}\mathcal{B}_{m,n}$  is not semisimple and seminormal representations may not exist. By contrast, Specht modules exist for all parameter values, are generically irreducible, and become reducible at special parameter values (playing a role somewhat similar to that played for Lie algebras by Verma modules). The  $\text{qw}\mathcal{B}_{m,n}$  Specht modules have been discussed in [33, 34, 35]. We construct them rather explicitly, by extending our “diagrammatic” view of the  $\text{qw}\mathcal{B}$  algebra to representations. A construction that combines categorial diagrams with Young tableaux is called the link-state representation here, to emphasize a similarity (or the authors’ prejudices regarding this similarity) to a link-state construction for the Temperley–Lieb algebra (see, e.g., [36, 37]).

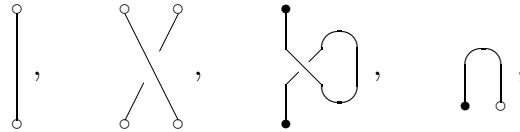
This paper is organized as follows. In Sec. 2, we introduce  $\text{qw}\mathcal{B}_{m,n}$  as an algebra of tangles of a particular type satisfying certain relations [7]. This language naturally suggests a construction for Jucys–Murphy elements. In Sec. 3, by “introducing spectral parameters into tangles,” we obtain a Baxterization (not of the algebra, but of the “ambient” category), which allows us to construct commutative families of algebra elements; the families depend on two parameters in addition to the  $\text{qw}\mathcal{B}_{m,n}$  parameters and are also determined by one out of three expansion points of the universal transfer matrix. In Sec. 4, we construct the  $\text{qw}\mathcal{B}_{m,n}$  Specht modules in terms of tangles in which the propagating lines (“defects”) end at the boxes of two standard Young tableaux. In Sec. 5, we construct the seminormal representations and find the spectrum of Jucys–Murphy elements.

## 2. $\text{qw}\mathcal{B}_{m,n}$ FROM BRAIDED MONOIDAL CATEGORIES

In this section, we define the quantum walled Brauer algebra in terms of diagrams (tangles); these can be thought of as being supplied by a rigid braided monoidal category  $\mathbb{B}_{\bullet,\circ}$ ; specifically, the category objects are given by (ordered) collections of nodes of two sorts ( $\bullet$  and  $\circ$ ) and morphisms are given by tangles on these nodes. Quantum walled Brauer algebras  $\text{qw}\mathcal{B}_{m,n}$  with  $m, n \in \mathbb{N}_0$  are endomorphism algebras of  $\mathbb{B}_{\bullet,\circ}$  objects; taking the abelianization  $\mathbb{A}_{\bullet,\circ}$  of  $\mathbb{B}_{\bullet,\circ}$  then gives a category whose simple objects are all simple  $\text{qw}\mathcal{B}_{m,n}$ -modules for  $m, n \in \mathbb{N}_0$ .

**2.1. The category  $\mathbb{B}_{\bullet,\circ}$ .** We fix a braided  $\mathbb{C}$ -vector space  $X$ . Objects in  $\mathbb{B}_{\bullet,\circ}$  are tensor products of  $X$  and its dual  $X^*$ . We use the respective notation  $\bullet$  and  $\circ$  for  $X^*$  and  $X$  and call them the black and white objects. Their tensor products are represented simply as  $\bullet\bullet = X^* \otimes X^*$ ,  $\bullet\circ\bullet = X^* \otimes X \otimes X^*$ , and so on.

**2.1.1. Diagram notation for morphisms.** We use the standard pictorial notation, representing morphisms as tangles [38] (cf. [7, 13]). Characteristic examples of morphisms are



which are the identity morphism of  $X$ , the braiding  $X \otimes X \rightarrow X \otimes X$ , the ribbon morphism on (or the twist of)  $X^*$ , and the coevaluation of  $X^*$  and  $X$ . All diagrams are considered modulo isotopy (more precisely, a tangle is an isotopy class of diagrams), and we assume that there are no triple intersections of lines.

**2.1.2. Relations in  $\text{Mor}(\mathbb{B}_{\bullet,\circ})$ .** The set of morphisms between two objects in  $\mathbb{B}_{\bullet,\circ}$  is a  $\mathbb{C}$ -vector space of formal linear combinations of tangles modulo the relations that we now describe and which depend on four complex parameters  $\alpha, \beta, \kappa, \kappa' \in \mathbb{C}$ .

The basic axiom is that the braiding of two white objects satisfies the Hecke relation

$$(2.1) \quad \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} = -\alpha\beta \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} | \\ | \\ | \end{array} + (\alpha + \beta) \begin{array}{c} \circ \quad \circ \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \circ \quad \circ \end{array}.$$

We also assume that the category is rigid, with the black and white objects being dual to each other; this means that there are the evaluation and coevaluation morphisms  $\begin{array}{c} \bullet \\ \cup \end{array}$ ,  $\begin{array}{c} \circ \\ \cap \end{array}$ ,  $\begin{array}{c} \bullet \\ \cap \end{array}$ , and  $\begin{array}{c} \circ \\ \cup \end{array}$ . Moreover, the category is assumed to be ribbon, with the ribbon map on black and white objects defined by two constants  $\kappa$  and  $\kappa'$ :

$$(2.2) \quad \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{\kappa} \begin{array}{c} | \\ | \\ | \end{array} \quad \text{and} \quad \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \end{array} = \frac{1}{\kappa'} \begin{array}{c} | \\ | \\ | \end{array}.$$

We note that by duality, these relations imply that

$$(2.3) \quad \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \bullet \end{array} = \frac{1}{\kappa} \begin{array}{c} | \\ | \\ | \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \circ \end{array} = \frac{1}{\kappa'} \begin{array}{c} | \\ | \\ | \end{array}.$$

Below, it is convenient to use these relations in the form

$$(2.4) \quad \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \kappa \begin{array}{c} \cap \\ \cap \end{array} \quad \text{and} \quad \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \end{array} = \kappa' \begin{array}{c} \cup \\ \cup \end{array}.$$

Composing (2.1) with evaluation and coevaluation maps, we readily obtain the Hecke-algebra relations for two black objects, the same as for the white ones:

$$(2.5) \quad \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = -\alpha\beta \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} | \\ | \\ | \end{array} + (\alpha + \beta) \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}.$$

Simple manipulations also show that

$$(2.6) \quad \begin{array}{c} \bullet \quad \circ \\ \diagdown \quad \diagup \\ \bullet \quad \circ \end{array} = -\frac{1}{\alpha\beta} \begin{array}{c} | \\ | \\ | \end{array} \begin{array}{c} | \\ | \\ | \end{array} + \frac{\alpha + \beta}{\alpha\beta} \kappa \begin{array}{c} \bullet \quad \circ \\ \diagup \quad \diagdown \\ \bullet \quad \circ \end{array}.$$

A similar formula holds with the colors interchanged and  $\kappa$  replaced with  $\kappa'$ .

Compositions with  $\begin{array}{c} \bullet \\ \cup \end{array}$  and  $\begin{array}{c} \circ \\ \cap \end{array}$  yield

$$(2.7) \quad \begin{array}{c} \bullet \\ \cup \end{array} \begin{array}{c} \circ \\ \cap \end{array} = \frac{\alpha\beta\kappa\kappa' + 1}{\kappa'(\alpha + \beta)} \quad \text{and} \quad \begin{array}{c} \circ \\ \cap \end{array} \begin{array}{c} \bullet \\ \cup \end{array} = \frac{\alpha\beta\kappa\kappa' + 1}{\kappa(\alpha + \beta)}.$$

**2.2. The algebra  $\text{qw}\mathcal{B}_{m,n}$  [7].** The quantum walled Brauer algebra  $\text{qw}\mathcal{B}_{m,n}$  is the algebra of endomorphisms of the object (“mixed tensor space”)

$$\mathcal{T}_{m,n} = (X^*)^{\otimes m} \otimes X^{\otimes n} = \underbrace{\bullet \cdots \bullet}_m \underbrace{\circ \cdots \circ}_n.$$

Each such endomorphism can be represented by a tangle, considered modulo the above relations, with  $m$  black and  $n$  white nodes on the top edge and the same numbers of black and white nodes on the bottom edge, and with strands connecting a white (black) node with a white (black) node on the opposite edge or with a black (white) node on the same edge. If a strand connects nodes on the same edge, we call it an arc (top or bottom depending on the edge); there must be the same number of top and bottom arcs. The strands connecting different edges are called defects<sup>3</sup> (black or white depending on the type of connected nodes).

A vertical *wall* can be imagined to separate  $(X^*)^{\otimes m} = \bullet \cdots \bullet$  from  $X^{\otimes n} = \circ \cdots \circ$ . Arcs necessarily cross the wall, while defects do not.<sup>4</sup>

**2.2.1. Numbering convention.** With the order of factors chosen as  $X^{*\otimes m} \otimes X^{\otimes n}$  in what follows, we adopt the convention that the nodes are enumerated “from the wall” outwards. We also sometimes use a primed collection of the integers for labeling the (“black”)  $X^*$  factors:

$$\begin{array}{cccccccc} \dots & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \dots \\ \dots & 4' & 3' & 2' & 1' & 1 & 2 & 3 & 4 & \dots \end{array}$$

**2.2.2. The  $\text{qw}\mathcal{B}_{m,n}$  generators and relations.** The tangles are multiplied standardly (in accordance with our convention of reading the diagrams from top down), with relations (2.1)–(2.7) applied whenever needed, and then the algebra  $\text{qw}\mathcal{B}_{m,n}$  is generated by the tangles [10]

$$(2.8) \quad g_j = \begin{array}{ccccccc} \bullet & \dots & \bullet & \dots & \bullet & \circ & \dots & \circ \\ | & & \diagdown & & | & | & & | \\ \dots & & \diagup & & \dots & \dots & & \dots \\ \bullet & \dots & \bullet & \dots & \bullet & \circ & \dots & \circ \\ m' & & j+1' & & j' & 1' & 1 & n \end{array}, \quad j = 1, \dots, m-1,$$

$$(2.9) \quad \mathcal{E} = \begin{array}{ccccccc} \bullet & \dots & \bullet & \bullet & \circ & \dots & \circ \\ | & & | & \cap & | & & | \\ \dots & & \dots & \cup & \dots & & \dots \\ \bullet & \dots & \bullet & \bullet & \circ & \dots & \circ \\ m' & & 2' & 1' & 1 & 2 & n \end{array}$$

<sup>3</sup>Propagating lines in another nomenclature in a similar context [5].

<sup>4</sup>For each pair of positive integers  $(m, n)$ , the algebra  $\text{qw}\mathcal{B}_{m,n}$  can also be represented as endomorphisms of an object in  $\mathbb{B}_{\bullet, \circ}$  where the  $m$  factors  $X^*$  and the  $n$  factors  $X$  are taken in a different order. All such algebras are isomorphic because of the existence of braiding morphisms in the category  $\mathbb{B}_{\bullet, \circ}$ . For example, in the case  $m = n, n \neq 1$ , there is a natural “physical” order of factors in the tensor product  $X \otimes X^* \otimes X \otimes X^* \otimes \dots$ , which represents a spin chain of alternating atoms of two sorts.

and

$$(2.10) \quad h_i = \begin{array}{ccccccc} \bullet & \dots & \bullet & \circ & \dots & \circ & \dots & \circ \\ | & & | & | & & \diagdown & \diagup & | \\ \bullet & \dots & \bullet & \circ & \dots & \circ & \dots & \circ \\ m' & & 1' & 1 & & i & i+1 & n \end{array}, \quad i = 1, \dots, n-1.$$

By Eqs. (2.1) and (2.5), the  $g_j$  and  $h_i$  are standard generators of two commuting Hecke algebras  $\mathcal{H}_m(\alpha, \beta)$  and  $\mathcal{H}_n(\alpha, \beta)$ ,

$$(2.11) \quad g_j^2 = (\alpha + \beta)g_j - \alpha\beta \cdot 1, \quad h_i^2 = (\alpha + \beta)h_i - \alpha\beta \cdot 1,$$

$$(2.12) \quad g_j h_i = h_i g_j,$$

where  $1 \leq j \leq m-1$  and  $1 \leq i \leq n-1$ . The other relations are as follows:

$$(2.13) \quad \mathcal{E}\mathcal{E} = \frac{\theta+1}{\kappa(\alpha+\beta)}\mathcal{E},$$

$$(2.14) \quad \mathcal{E}g_1\mathcal{E} = \frac{1}{\kappa}\mathcal{E}, \quad \mathcal{E}h_1\mathcal{E} = \frac{1}{\kappa}\mathcal{E},$$

$$(2.15) \quad \mathcal{E}g_j = g_j\mathcal{E}, \quad 2 \leq j \leq m-1, \quad \mathcal{E}h_i = h_i\mathcal{E}, \quad 2 \leq i \leq n-1,$$

$$(2.16) \quad \mathcal{E}g_1h_1^{-1}\mathcal{E}(g_1 - h_1) = 0, \quad (g_1 - h_1)\mathcal{E}g_1h_1^{-1}\mathcal{E} = 0.$$

Here,

$$\theta = \alpha\beta\kappa\kappa'.$$

**2.2.3.** Relations (2.13)–(2.16) follow immediately from the diagram representation of the generators. In particular, it is obvious that  $\mathcal{E}$  commutes with  $g_{\geq 2}$  and  $h_{\geq 2}$ . Next, relations (2.14), which take the form

$$\begin{array}{c} \bullet \quad \circ \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ \bullet \quad \circ \end{array} = \frac{1}{\kappa} \begin{array}{c} \bullet \quad \circ \\ | \quad | \\ \bullet \quad \circ \end{array}, \quad \begin{array}{c} \bullet \quad \circ \\ | \quad | \\ \diagup \quad \diagdown \\ | \quad | \\ \bullet \quad \circ \end{array} = \frac{1}{\kappa} \begin{array}{c} \bullet \quad \circ \\ | \quad | \\ \bullet \quad \circ \end{array},$$

are corollaries of the  $\kappa$ -relation in (2.2)–(2.4). As regards relations (2.16), we note that, graphically,

$$(2.17) \quad \Omega := \mathcal{E}g_1h_1^{-1}\mathcal{E} = \begin{array}{c} \bullet \quad \circ \\ | \quad | \\ \diagdown \quad \diagup \\ | \quad | \\ \bullet \quad \circ \end{array}$$

whence it is obvious that  $\Omega g_1$  is the same as  $\Omega h_1$  up to isotopy, and equivalently for  $g_1\Omega$  and  $h_1\Omega$ .

The two relations in (2.16) can be equivalently rewritten with  $g_1h_1^{-1}$  replaced by  $g_1^{-1}h_1$ .

**2.2.4. The algebra parameters.** The algebra relations involve the parameters  $\alpha$ ,  $\beta$ ,  $\kappa$ , and  $\theta$ , and we sometimes write  $\text{qwB}_{m,n}(\alpha, \beta, \kappa, \theta)$  for the algebra, although two parameters can be eliminated from the relations by rescaling the generators.

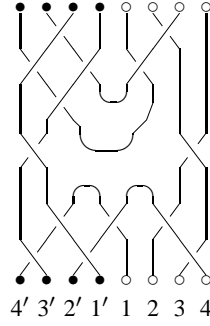


FIGURE 1. A reduced tangle representing an endomorphism of  $(X^*)^{\otimes 4} \otimes X^{\otimes 4}$ . The tangle contains two top and two bottom arcs.

**2.3. Reduced tangles.** Using relations (2.1)–(2.7), each tangle can be rewritten as a linear combination of tangles of special form, which we call reduced tangles. A reduced tangle is a tangle in which

- (1) no strand crosses itself,
- (2) every two strands cross at most once (in particular, top arcs do not cross bottom arcs),
- (3) there are no loops,
- (4) and the following preference rules hold for all other crossings:
  - (a) A strand connected to a black node overcrosses any strand not connected to a black node.
  - (b) A defect or bottom arc connected to a bottom black node with a number  $a'$  (with convention **2.2.1**) overcrosses any defect or bottom arc connected to a bottom black node with a number  $b' < a'$ .
  - (c) Any black defect overcrosses any top arc.
  - (d) For top arcs, the same convention is applied as for the bottom arcs, based on the number of black ends of the arcs (also counted outward from the wall): the arc with a smaller number of the black end overcrosses.
  - (e) For two white defects, the one connected to the  $a$ th bottom node overcrosses that connected to the  $b$ th bottom node if and only if  $a > b$ .

An example of a reduced tangle is shown in Fig. 1.<sup>5</sup>

The overcrossing preferences set for reduced tangles can be equivalently expressed in terms of a *drawing order*. Imagine a tangle drawn in ink; lines are drawn one by one, and a new line breaks each time it meets a line already drawn, which means a new line undercrosses every old one. Then, first, lines connected to the bottom black dots  $1', 2', \dots$

<sup>5</sup>Somewhat different conventions, with the same effect, are used in [39] (the terminology also differs in one essential point: our defects and arcs are all called arcs there).

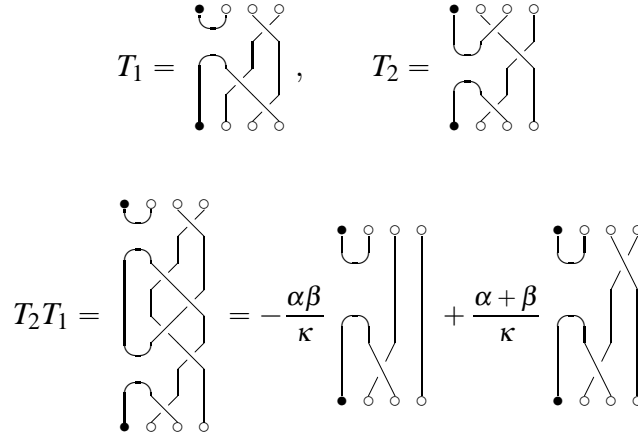


FIGURE 2. Reduced tangles  $T_1$  and  $T_2$ , their product as a nonreduced tangle, and the product expressed in terms of reduced tangles.

are drawn in this order. Next, top arcs are drawn in the ascending order  $b'_1 < b'_2 < \dots$  of their black ends. Finally, the white defects are drawn in the *descending* order  $w_1 > w_2 > \dots$  of their bottom ends.

The reduced tangles form a basis in  $\text{qw}\mathcal{B}_{m,n}$ .

Multiplication of reduced tangles is defined in a standard way: for two reduced tangles  $T_1$  and  $T_2$ , the product  $T_1 T_2$  is the tangle obtained by placing  $T_1$  under  $T_2$ , identifying the nodes on the top edge of  $T_1$  with those on the bottom edge of  $T_2$  according to their numbers, and then removing the intermediate nodes. The resultant tangle is not reduced in general and should be rewritten as a linear combination of reduced tangles using relations (2.1)–(2.7).

An example of calculating the product of two reduced tangles is given in Fig. 2.

**2.4. Jucys–Murphy elements.** We use the diagram representation to define a family of Jucys–Murphy elements for  $\text{qw}\mathcal{B}_{m,n}$ :  $J(n)_1 = 1, J(n)_2, \dots, J(n)_{m+n}$ . We first define  $J(n)_2, \dots, J(n)_n$  just as the Jucys–Murphy elements for the “white” Hecke algebra  $\mathcal{H}_n(\alpha, \beta)$  (see B.2):

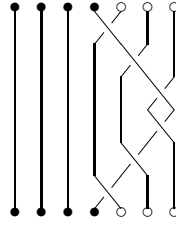
$$(2.18) \quad J(n)_1 = 1, \quad J(n)_i = (-\alpha\beta)^{-1} h_{n+1-i} J(n)_{i-1} h_{n+1-i}, \quad i = 2, \dots, n.$$

In particular,

$$J(n)_n = (-\alpha\beta)^{-n+1} \left[ \text{Diagram} \right]$$



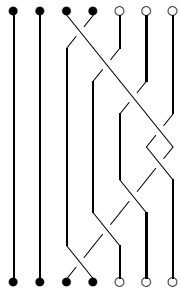
(with the total number  $n$  of white strands). The higher elements  $J(n)_{n+1}, \dots, J(n)_{m+n}$  are defined similarly in graphical terms, but quite differently in essence:

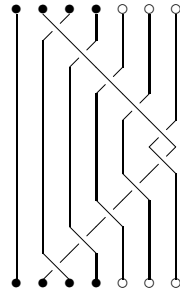
$$J(n)_{n+1} = (-\alpha\beta)^n$$


and then

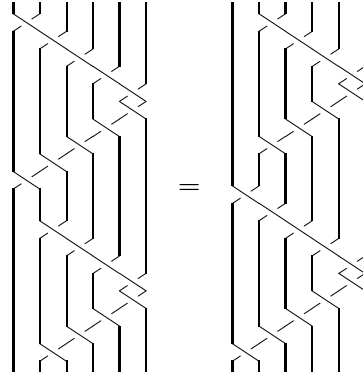
$$J(n)_{n+j} = (-\alpha\beta)^{-1} g_{j-1} J(n)_{n+j-1} g_{j-1}, \quad j \geq 2,$$



that is,

$$J(n)_{n+2} = (-\alpha\beta)^{n-1}$$


$$J(n)_{n+3} = (-\alpha\beta)^{n-2}$$


and so on. All the  $J(n)_j$  thus defined pairwise commute, simply because (cf. [23])



irrespective of the color of the lines, just by virtue of the braid relations. But the definition of  $J(n)_{n+1}, \dots, J(n)_{m+n}$  involves morphisms from the braided tensor category,  and , which are not  $\text{qwB}$  elements.

**2.4.1. Lemma.** *All  $J(n)_j$ ,  $j = 2, \dots, m+n$ , are elements of  $\text{qwB}_{m,n}$ .*

This follows by noting the recursion relations

$$J(n)_{n+j} = J(n-1)_{n+j-1} - \kappa(\alpha + \beta) U(j, n), \quad j \geq 2,$$

with

$$U(j, n) = (-\alpha\beta)^{-j+n} g_{j-1,1}^{\searrow} h_{n-1,1}^{-1 \searrow} \mathcal{E} h_{1,n-1}^{-1 \swarrow} g_{1,j-1}^{\swarrow}$$

where we define the “contiguous” products of generators

$$(2.19) \quad \begin{aligned} g_{m,n}^{\nearrow} &= g_m g_{m+1} \cdots g_n, & g_{m,n}^{-1 \nearrow} &= g_m^{-1} g_{m+1}^{-1} \cdots g_n^{-1}, \\ g_{m,n}^{\searrow} &= g_m g_{m-1} \cdots g_n, & g_{m,n}^{-1 \searrow} &= g_m^{-1} g_{m-1}^{-1} \cdots g_n^{-1} \end{aligned}$$

(no inversion of the order of factors in the second definition), and similarly for  $h_{m,n}$ . The  $U(j, n)$  are elements of  $\text{qwB}_{m,n}$  (by the embeddings of the “lower”  $\text{qwB}$  algebras), and so is the initial condition, which is given just by the right-hand side of (2.6) times  $-\alpha\beta$ :

$$J(1)_2 = \text{id} - (\alpha + \beta)\kappa \mathcal{E}.$$

**2.4.2.** Solving these recursion relations, we find the higher Jucys–Murphy elements explicitly:

$$J(n)_{n+j} = (-\alpha\beta)^{1-j} g_{j-1,1}^{\searrow} \left( 1 - (\alpha + \beta)\kappa \sum_{i=1}^n (-\alpha\beta)^{i-1} h_{i-1,1}^{-1 \searrow} \mathcal{E} h_{1,i-1}^{-1 \nearrow} \right) g_{1,j-1}^{\nearrow}.$$

**2.5. Casimir elements.** We continue using diagram representations to obtain some special  $\text{qwB}$  elements.

**2.5.1.** We define a central element  $C_{m,n} \in \text{qwB}_{m,n}$ , which we call a Casimir element, as

$$\kappa(\alpha + \beta)C_{m,n} = \begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array}$$

where the two strands with larger endpoints respectively represent the bunches of  $m$  black and  $n$  white strands. That  $C_{m,n}$  is central is obvious from this representation. That it is an element of  $\text{qwB}_{m,n}$  follows from the easily established recursion relations

$$\begin{aligned} C_{j,n} &= -\alpha\beta C_{j-1,n} - \frac{1}{\theta} (-\alpha\beta)^{j-n} J(n)_{n+j}, \quad j \geq 1, \\ C_{0,n} &= -\frac{1}{\alpha\beta} C_{0,n-1} + \frac{1}{\alpha\beta} h_{n-1,1}^{-1 \searrow} h_{1,n-1}^{-1 \nearrow}, \quad n \geq 2, \\ C_{0,1} &= \omega \cdot 1, \end{aligned}$$

where  $\omega$  can be chosen arbitrarily (the above diagram defines  $\omega = \frac{1}{\alpha\beta} \left( 1 - \frac{1}{\kappa(\alpha + \beta)} \right)$ , but introducing an arbitrary constant does not affect the property of  $C_{m,n}$  to be central).

**2.5.2.** Solving these recursion relations, we find

$$\begin{aligned} C_{m,n} = C_{m,n}(\omega) &= -\frac{1}{\theta} (-\alpha\beta)^{m-n} \sum_{j=n+1}^{n+m} J(n)_j - \sum_{i=1}^{n-1} (-\alpha\beta)^{m-i} h_{i,n-1}^{-1 \nearrow} h_{n-1,i}^{-1 \searrow} \\ &\quad + \omega (-\alpha\beta)^{m+1-n} \cdot 1. \end{aligned}$$

**2.5.3.** Another central element is obviously given by

$$\kappa'(\alpha + \beta) \tilde{C}_{m,n}(\tilde{\omega}) = \text{Diagram},$$

which is again defined as a  $\text{qwB}$  element by appropriate recursion relations, with the boundary condition  $\tilde{C}_{1,0}(\tilde{\omega}) = \tilde{\omega} \cdot 1$ , where  $\tilde{\omega}$  can also be chosen arbitrarily.

### 3. BAXTERIZATION AND COMMUTING FAMILIES

It is well known that the relations in the Hecke algebra  $\mathcal{H}_m(\alpha, \beta)$  (see Appendix B for the conventions) can be equivalently stated as the “Yang–Baxter equations with a spectral parameter,”

$$(3.1) \quad g_i(w)g_{i+1}(zw)g_i(z) = g_{i+1}(z)g_i(zw)g_{i+1}(w),$$

for

$$g_i(z) = g_i + \frac{\alpha + \beta}{z - 1} \cdot 1, \quad z \in \mathbb{C}.$$

We now propose a similar construction for  $\text{qwB}_{m,n}$ : namely, we “Baxterize” the morphisms  $g = \text{Diagram}$  and  $h = \text{Diagram}$ , as well as  $\text{Diagram}$  and  $\text{Diagram}$ , even though the last two are not elements of  $\text{qwB}$ ; but using them allows constructing commuting families of  $\text{qwB}$  elements modeled on conservation laws in integrable systems of statistical mechanics.

**3.1. Baxterized morphisms.** In the language of diagrams, we write the above  $g_i(z)$  as

$$g(z) = \text{Diagram} = \text{Diagram} + \frac{\alpha + \beta}{z - 1} \text{Diagram}.$$

The Yang–Baxter equation with a spectral parameter, Eq. (3.1), is then standardly expressed as

$$\text{Diagram} = \text{Diagram},$$

The same of course holds for white lines, with

$$h(z) = \text{Diagram} = \text{Diagram} + \frac{\alpha + \beta}{z - 1} \text{Diagram}.$$

In the setting of the braided category in **2.1**, we now extend these definitions to the “mixed” cases. The following lemma is easy to show by direct verification.

**3.1.1. Lemma.** *Let*

$$\begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} \begin{array}{c} \circ \\ \diagup \\ \bullet \end{array} z = \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} + \frac{\alpha + \beta}{\alpha\beta} \frac{u}{z - u} \begin{array}{c} \circ \\ \diagup \\ \circ \end{array} \begin{array}{c} \circ \\ \diagdown \\ \circ \end{array}, \quad \begin{array}{c} \circ \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array} z = \begin{array}{c} \circ \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array} + \frac{\alpha + \beta}{\alpha\beta} \frac{v}{z - v} \begin{array}{c} \circ \\ \diagup \\ \circ \end{array} \begin{array}{c} \circ \\ \diagdown \\ \circ \end{array}$$

with parameters  $u$  and  $v$ . Then all “mixed” Yang–Baxter equations with spectral parameter (the equations involving black and white lines, such as

$$\begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} \begin{array}{c} \circ \\ \diagup \\ \bullet \end{array} z \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} zw = \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \circ \\ \diagup \\ \bullet \end{array} zw \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} z$$

and others) hold if

$$uv = -\theta.$$

We assume the last equation to hold in what follows.

**3.1.2.** We note that

$$\begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} \begin{array}{c} \circ \\ \diagup \\ \bullet \end{array} z \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} w = -\frac{1}{\alpha\beta} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} + \frac{\kappa(\alpha + \beta)}{\alpha\beta} \frac{zw - 1}{(z - u)(w - v)} \begin{array}{c} \circ \\ \diagup \\ \circ \end{array} \begin{array}{c} \circ \\ \diagdown \\ \circ \end{array}.$$

Hence, first, the left-hand side is an element of  $q\mathcal{WB}_{m,n}(z, w) = q\mathcal{WB}_{m,n} \otimes \mathbb{C}(z, w)$  and, second, setting  $w = 1/z$  yields a pair of morphisms that are essentially inverse to each other.

## 3.2. “Universal transfer matrix” and conservation laws.

**3.2.1.** Following **2.5.1** in spirit, but using the Baxterized operations introduced above, we define

$$\kappa(\alpha + \beta)\mathcal{A}_{m,n}(z, w) = \begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} \begin{array}{c} \circ \\ \diagup \\ \bullet \end{array} z \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array} w \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array}.$$

The black and white strands with larger endpoints respectively represent  $m$  black and  $n$  white strands. We emphasize that the diagram contains *three* types of Baxterized operations: traveling from top down along the right-hand part of the loop, we first encounter  $\begin{array}{c} \bullet \\ \diagdown \\ \circ \end{array} \begin{array}{c} \circ \\ \diagup \\ \bullet \end{array} z$  repeated  $m$  times, then  $\begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \bullet \end{array}$  repeated  $n$  times, and then  $\begin{array}{c} \circ \\ \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \\ \circ \end{array} w$  repeated also  $n$  times;



where, first, the big circles represent  $m$  black and  $n - 1$  white strands and, second, we define

$$\begin{aligned}\tilde{h}(z) &:= \text{diagram} = \text{diagram} + \frac{(\alpha + \beta)u}{\alpha\beta} \frac{1}{z - u} \text{diagram}, \\ \tilde{\tilde{h}}(z) &:= \text{diagram} = \text{diagram} + \frac{(\alpha + \beta)v}{\alpha\beta} \frac{1}{z - v} \text{diagram}.\end{aligned}$$

We need only  $\tilde{\mathcal{J}}_{0,n}(z, w)$  in (3.5). Because  $\tilde{\mathcal{J}}_{0,1}(z, w) = 1$ , we conclude that

$$\tilde{\mathcal{J}}_{0,n}(z, w) = (\tilde{h}_{n-1}(w) \dots \tilde{h}_1(w)) (\tilde{h}_1(z) \dots \tilde{h}_{n-1}(z)),$$

which is an element of  $\text{qw}\mathcal{B}_{0,n}(z, w) \subset \text{qw}\mathcal{B}_{m,n}(z, w)$ .

We return to elements (3.4), which appear in recursion relations (3.3). Clearly,

$$(3.7) \quad \mathcal{J}_{m,n}(z, w) = -\frac{1}{\alpha\beta} g_{m-1}(w) \mathcal{J}_{m-1,n}(z, w) g_{m-1}(z), \quad m \geq 2.$$

It therefore remains to calculate  $\mathcal{J}_{1,n}(z, w)$ . Once again by the identity in 3.1.2,

$$(3.8) \quad \mathcal{J}_{1,n}(z, w) = \mathcal{J}_{1,n-1}(z, w) + \kappa(\alpha + \beta)(-\alpha\beta)^{n-1} \frac{1 - zw}{(z - u)(w - v)} \mathcal{U}_{1,n}(z, w),$$

where

$$\mathcal{U}_{1,n}(z, w) = \tilde{h}_{n-1}(w) \dots \tilde{h}_1(w) \mathcal{E} \tilde{h}_1(z) \dots \tilde{h}_{n-1}(z).$$

The initial condition is  $\mathcal{J}_{1,0}(z, w) = 1$ .

This finishes the proof because recursion relations (3.3), (3.5), (3.7), and (3.8) define  $\mathcal{A}_{m,n}(z, w)$  as an element of  $\text{qw}\mathcal{B}_{m,n}(z, w)$ .

**3.2.3. Remark.** We note the limits

$$\lim_{z, w \rightarrow \infty} \mathcal{J}_{m,n}(z, w) = J(n)_{m+n}$$

with a Jucys–Murphy element in the right-hand side, and

$$\lim_{z, w \rightarrow \infty} \mathcal{A}_{m,n}(z, w) = \tau \cdot 1 + C_{m,n},$$

where  $C_{m,n}$  is defined in 2.5.1, and  $\tau$  can be made equal to zero by choosing  $\omega$  in 2.5.2 as

$$\omega = \frac{\alpha^2 \theta + \alpha \beta \theta - \alpha \beta + \beta^2 \theta}{\alpha \beta \theta (\alpha + \beta)^2}.$$

**3.2.4. “Universal” transfer matrix.** For a fixed  $\rho \in \mathbb{C}$ , we set

$$\mathcal{A}_{m,n}(z) = \mathcal{A}_{m,n}(z, \rho z).$$

When  $\text{qw}\mathcal{B}_{m,n}$  acts on a particular lattice model, this object is a transfer matrix.<sup>6</sup> By

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<sup>6</sup>And  $\mathcal{J}_{m,n}(z, \rho z)$  in (3.4), the monodromy matrix.

extension, we call  $\mathcal{A}_{m,n}(z)$  the transfer matrix or, to emphasize its independence from a particular lattice model, the universal transfer matrix.

**3.3. Theorem.**  $\mathcal{A}_{m,n}(z)$  is a generating function for a commutative family of elements of  $qWB_{m,n}$ :

$$\mathcal{A}_{m,n}(z)\mathcal{A}_{m,n}(w) - \mathcal{A}_{m,n}(w)\mathcal{A}_{m,n}(z) = 0.$$

The proof is by the (generally standard, but here slightly more involved) use of the “train argument” [40] (i.e., the Yang–Baxter equation with the spectral parameter).

**3.4. Solving the recursion relations.** Similarly to 2.4.2, we can solve the above recursion relations to find a relatively explicit expression for  $\mathcal{J}_{m,n}(z, w)$  and  $\mathcal{A}_{m,n}(z, w)$ .

**3.4.1. Lemma.** *We have*

$$\begin{aligned} \mathcal{J}_{m,n}(z, w) &= (-\alpha\beta)^{-m+1} g_{m-1}(w) \dots g_1(w) \\ &\quad \left( 1 + \frac{(1-zw)(\alpha+\beta)\kappa}{(z-u)(w-v)} \sum_{s=1}^n (-\alpha\beta)^{s-1} \tilde{h}_{s-1}(w) \dots \tilde{h}_1(w) \mathcal{E} \tilde{h}_1(z) \dots \tilde{h}_{s-1}(z) \right) \\ &\quad g_1(z) \dots g_{m-1}(z). \end{aligned}$$

We note that

$$\mathcal{J}_{m,n}(z, \frac{1}{z}) = \frac{(z\alpha + \beta)^{m-1}(\alpha + z\beta)^{m-1}}{(z-1)^{2m-2}(\alpha\beta)^{m-1}} \cdot 1.$$

**3.4.2. Lemma.** *The universal transfer matrix is given by*

$$\begin{aligned} \mathcal{A}_{m,n}(z, w) &= \xi_1(z, w) \cdot 1 - \frac{zw + \theta}{\theta(z-1)(w-1)} (-\alpha\beta)^{m-n} \sum_{j=1}^m \mathcal{J}_{j,n}(z, w) \\ &\quad + \frac{1-zw}{(z-u)(w-v)} \sum_{i=1}^{n-1} (-\alpha\beta)^{m-i} \bar{\mathcal{J}}_{i,n-1}(z, w), \end{aligned}$$

where  $\xi_1(z, w)$  is a rational function of  $z$  and  $w$ , and

$$\bar{\mathcal{J}}_{i,n-1}(z, w) = (\tilde{h}_i(z) \dots \tilde{h}_{n-1}(z)) (\tilde{h}_{n-1}(w) \dots \tilde{h}_i(w)).$$

**3.5. Expanding the transfer matrix.** It follows that  $\bar{\mathcal{J}}_{i,n-1}(z, -\theta/z)$  is proportional to the identity,<sup>7</sup> and hence the second sum in the formula for  $\mathcal{A}_{m,n}(z, w)$  has the form  $\xi_2(z, w) \cdot 1 + (zw + \theta)A_2(z, w)$ . Combining this with the structure of  $\mathcal{J}_{m,n}(z, w)$  in 3.4.1, we conclude that

$$\mathcal{A}_{m,n}(z, w) = \xi(z, w) \cdot 1 - (zw + \theta)(1 - zw)\mathcal{A}_{m,n}(z, w),$$

---

<sup>7</sup>In fact,  $\bar{\mathcal{J}}_{i,n-1}(z, -\theta/z) = (-1)^{n-i} \frac{(z\alpha + u\beta)^{n-i}(u\alpha + z\beta)^{n-i}}{(z-u)^{2(n-i)}(\alpha\beta)^{2(n-i)}} \cdot 1.$

with a rational function  $\xi(z, w)$  and with some  $A_{m,n}(z, w) \in \text{qw}\mathcal{B}_{m,n}(z, w)$  (regular at  $w = 1/z$  and  $w = -\theta/z$ ). This formula suggests two natural points around which the transfer matrix can be expanded to produce commuting “conservation laws” (“Hamiltonians”).<sup>8</sup> We comment on the expansion around one of these; to avoid square roots in the formulas, it is convenient to define the transfer matrix as a function of a single spectral parameter as  $\mathcal{A}_{m,n}(z) = \mathcal{A}_{m,n}(z, \rho^2 z)$ . Then

$$\mathcal{A}_{m,n}(z, \rho^2 z) = (\dots) \cdot 1 + \left(z - \frac{1}{\rho}\right) 2H_{m,n}^{(1)}(\rho) + \left(z - \frac{1}{\rho}\right)^2 H_{m,n}^{(2)}(\rho) + \dots$$

with a commutative family of elements  $H_{m,n}^{(j)}(\rho)$ ,  $j \geq 1$ . It then follows from the formulas in 3.4.2 and 3.4.1 that the first Hamiltonian is

$$\begin{aligned} H_{m,n}^{(1)}(\rho) = & -\frac{\rho^2}{(1-\rho u)(\rho-v)} \sum_{i=1}^{n-1} (-\alpha\beta)^{m-i} \bar{\mathcal{D}}_{i,n-1}\left(\frac{1}{\rho}, \rho\right) \\ & - \frac{(\alpha+\beta)(\theta+1)\rho^3}{\theta(\rho-1)^2(1-\rho u)(\rho-v)} \kappa \sum_{i=1}^m \sum_{j=1}^n (-\alpha\beta)^{m-n-i+j} \mathcal{E}_{i,j}\left(\frac{1}{\rho}, \rho\right) \\ & - \sum_{k=2}^m \sum_{j=1}^{k-1} (-1)^{j-k+1} (-\alpha\beta)^{m-k+1-n} \frac{\rho^3(\beta+\alpha\rho)^{k-1-j}(\alpha+\beta\rho)^{k-1-j}}{\theta(\rho-1)^{2(1-j+k)}} \\ & \times g_{k-1}(\rho) \dots g_{k-j+1}(\rho) g_{k-j}\left(\frac{1}{\rho}\right) \dots g_{k-1}\left(\frac{1}{\rho}\right), \end{aligned}$$

where

$$\mathcal{E}_{i,j}(z, w) = \tilde{h}_{j-1}(w) \dots \tilde{h}_1(w) g_{i-1}(w) \dots g_1(w) \mathcal{E} g_1(z) \dots g_{i-1}(z) \tilde{h}_1(z) \dots \tilde{h}_{j-1}(z).$$

The  $H_{m,n}^{(a)}(\rho)$ ,  $a \geq 1$ , depend on  $\rho$  and the chosen parameter  $u$ , in addition to the parameters of the algebra (see 2.2.4); the last formula applies in the case where  $\rho \neq 1/u$ ,  $\rho \neq -\theta/u$ , and  $\rho \neq 1$ ; at these special values, the expansion is rearranged.

#### 4. qwB SPECHT MODULES

We now extend the diagram presentation for  $\text{qw}\mathcal{B}_{m,n}$  to its modules. For generic values of the parameters  $\alpha, \beta, \kappa, \theta$ , the simple  $\text{qw}\mathcal{B}_{m,n}$  modules are labeled by pairs of Young diagrams  $(\lambda', \lambda)$  such that  $m - |\lambda'| = n - |\lambda|$  [33, 34]. We now construct  $\text{qw}\mathcal{B}_{m,n}$  Specht modules, which are the irreducible representations at generic parameters, but exist for all parameter values (and duly become reducible).<sup>9</sup> (A somewhat implicit description of

<sup>8</sup>A third possibility is, with  $w = \rho z$ , to take  $z \rightarrow \infty$ . Then the zeroth-degree term  $H_{m,n}^{(0)}$  in the expansion  $\mathcal{A}_{m,n}(z, \rho z) = H_{m,n}^{(0)} + \frac{1}{z} H_{m,n}^{(1)} + \dots$  is not proportional to the identity, but is *central* (see 3.2.3), and the first Hamiltonian may have to be defined as  $(H_{m,n}^{(0)})^{-1} H_{m,n}^{(1)}$ .

<sup>9</sup>In terms of a systematically categorical treatment, the modules must follow by passing from the category  $\mathbb{B}_{\bullet, \circ}$ , which is not abelian, to its abelianization  $\mathbb{A}_{\bullet, \circ}$  [41]. The objects of  $\mathbb{A}_{\bullet, \circ}$  are functors from  $\mathbb{B}_{\bullet, \circ}$  to the category of  $\mathbb{C}$ -vector spaces, and the morphisms are natural transformations between functors. The category  $\mathbb{B}_{\bullet, \circ}$  is then identified with a subcategory in  $\mathbb{A}_{\bullet, \circ}$  by the Yoneda functor. The monoidal structure in  $\mathbb{A}_{\bullet, \circ}$  can



Specht modules as subquotients of the regular  $\text{qw}\mathcal{B}_{m,n}$  module was given in [33].)

These modules are realized here as link-state representations, somewhat analogous to those for the Temperley–Lieb algebra (see, e.g., [37] and the references therein). Compared to the Temperley–Lieb case, where the construction is in terms of nonintersecting arcs only, we here have intersecting arcs and defect lines, as well as Young tableaux; the Young tableaux turn out to be “targets” for the defect lines. Informally, the construction of link states for  $\text{qw}\mathcal{B}_{m,n}$  can be summarized as follows: these are tangles made of (bottom) arcs and of defect lines that end at Young tableaux and serve to “propagate” the action of Hecke subalgebras to the tableaux (actually, Specht modules over the Hecke algebras);<sup>10</sup> the rules derived from those in 2.1.2 apply to disentangling arcs from each other and defects from arcs.

**4.1. Link states.** We fix  $m, n$ , and two Young diagrams  $\lambda'$  and  $\lambda$  (which can be empty),<sup>11</sup> with  $f := m - |\lambda'| = n - |\lambda| \geq 0$ , and describe a basis in the corresponding Specht module  $\mathcal{S}^{m,n,\lambda',\lambda}$ .

**4.1.1.** Basis vectors in  $\mathcal{S}^{m,n,\lambda',\lambda}$  are *link states*

$$|\ell' \xrightarrow{\chi} \ell, \mathfrak{t}', \mathfrak{t}\rangle,$$

where  $\ell' = (a'_1, \dots, a'_f) \subset (1, \dots, m)$ ,  $\ell = (a_1, \dots, a_f) \subset (1, \dots, n)$ , and  $\chi$  is a bijection, and  $\mathfrak{t}'$  and  $\mathfrak{t}$  are standard Young tableaux built on respective diagrams  $\lambda'$  and  $\lambda$ . It is convenient to say that  $\lambda', \mathfrak{t}'$ , etc., are *black*, and  $\lambda, \mathfrak{t}$ , etc., are *white*.

A link state uniquely determines a tangle of a special form as follows (see Fig. 3 for an example):

- (1) The tangle has  $m$  black and  $n$  white nodes in the bottom row.
- (2)  $\ell'$  and  $\ell$  list the numbers of (respectively black and white) bottom-row nodes supporting arcs, and the bijection identifies pairs of nodes connected by arcs.
- (3) The remaining  $m - f$  black nodes are mapped into the entries of  $\mathfrak{t}'$  strictly monotonically (and hence bijectively); we can say that *defect lines*, whose upper ends are associated with boxes of  $\mathfrak{t}'$ , do the mapping (see Fig. 3).

Similarly, the remaining  $n - f$  white nodes are connected to defect lines that determine their monotonic map into the entries of  $\mathfrak{t}$ .

be introduced following [42]. The category  $\mathbb{A}_{\bullet\circ}$  also admits an explicit description in terms of the  $\text{qw}\mathcal{B}_{m,n}$  representation categories for all  $m, n \in \mathbb{N}_0$ , as described in [14, Ch. 10] and [15]. For generic values of the parameters, Specht modules give all simple objects of  $\mathbb{A}_{\bullet\circ}$ .

<sup>10</sup>This has evident similarities with the construction in [5], where Specht modules of the walled Brauer algebra were constructed as  $\Delta_{m,n}(\lambda', \lambda) \cong I_{m,n}^f \otimes_{\Sigma_{m-f,n-f}} (S^{\lambda'} \boxtimes S^{\lambda})$ , where  $\Sigma_{m-f,n-f}$  is the product of two symmetric groups,  $S^{\lambda'}$  and  $S^{\lambda}$  are their Specht modules, and  $I_{m,n}^f$  is a space of “configurations of arcs.”

<sup>11</sup>For example, the fundamental objects  $X^*$  and  $X$  are particular cases of Specht modules:  $X^* = \mathcal{S}^{1,0,\square,\emptyset}$  and  $X = \mathcal{S}^{0,1,\emptyset,\square}$ .

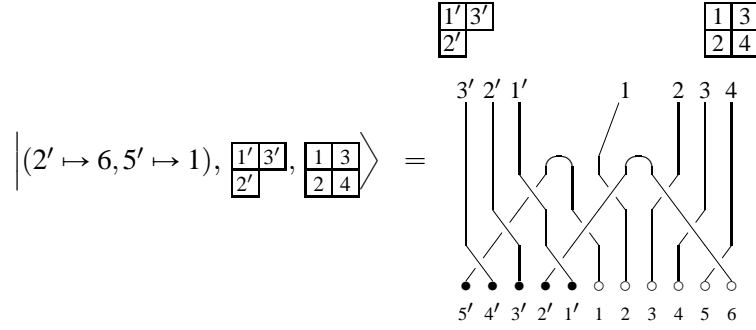


FIGURE 3. A link state for  $m = 5$ ,  $n = 6$ , and the two standard Young tableaux as indicated. There are  $f = 2$  arcs and hence three black and four white defects. Numbers at the upper ends of the defects show the tableau entries into which the defects are mapped. The maps  $(1' \mapsto 1', 3' \mapsto 2', 4' \mapsto 3')$  and  $(2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 3, 5 \mapsto 4)$  from bottom-edge nodes to the entries of the tableaux are monotonic, and hence the defects do not cross.

- (4) the crossing preference rules (inherited from those in 2.3) apply:
  - (a) black defects overcross arcs;
  - (b) an arc attached to a black node  $b$  overcrosses any arc attached to a black node  $c > b$ ;
  - (c) arcs overcross white defects; and
- (5) defect lines do not cross.

**4.1.2.  $\text{qw}\mathcal{B}_{m,n}$  action on links states.** The action of  $\text{qw}\mathcal{B}_{m,n}$  on link states is given by attaching the tangle corresponding to an element of  $\text{qw}\mathcal{B}_{m,n}$  to the bottom of the link-state tangle and forgetting the intermediate nodes. The resultant tangle is not necessarily a link state because it is not reduced. Each such nonreduced tangle can be rewritten as a linear combination of link states using relations (2.1)–(2.7) and the additional convention that a free arc connecting two defects vanishes,

$$(4.1) \quad \overbrace{m-f \dots k' \dots 1' \quad 1 \dots j \dots n-f}^{\text{free arc}} = 0 \quad (\text{for free arcs only}).$$

A free arc is one that is not linked with any other arc. (A linked arc can always be unlinked by using relations (2.1)–(2.7).)

We now apply the above rules to describe the action with the  $\text{qw}\mathcal{B}_{m,n}$  generators  $g_j$ ,  $\mathcal{E}$ , and  $h_i$  in more detail. After attaching the corresponding tangle in (2.8)–(2.10) to the bottom of the link state, the following reduction steps are needed to produce a linear combination of link states.

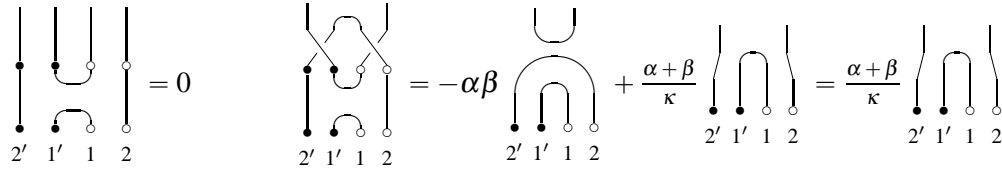


FIGURE 4. The result of applying  $\mathcal{E}$  to a link state with no arcs is zero, as shown on the left, because in this case the “upper arc” of  $\mathcal{E}$  is free (not linked with any arc). But whenever the original link state contains arc(s), the new arc coming from the action of  $\mathcal{E}$  is necessarily linked, with an example shown in the middle; disentangling the arc produces a number of terms, in one of which a free arc connects two defects, which gives zero.

**Acting with  $\mathcal{E}$ :** (1) if node  $1'$  is connected to an arc and node  $1$  is connected to a

defect, apply (2.3) to the resulting configuration

(2) if node  $1'$  is connected to a defect and node  $1$  is connected to an arc, apply

(2.2) to the resulting configuration

(3) if both nodes  $1'$  and  $1$  are connected to arcs, apply (2.4) to the resulting

configuration

(4) if both nodes  $1$  and  $1'$  are connected to defects, the result is zero if the tangle has no arcs, and is evaluated using the algebra relations for disentangling linked arcs and then applying (4.1) to a free arc (see Fig. 4).

**Acting with  $g_{j'}$ :** (1) if black node  $j'$  is connected to a defect and black node  $(j +$

$1)'$  is connected to an arc, then use (2.5) in the resulting configuration

[If node  $j'$  is connected to an arc and node  $(j + 1)'$  is connected to a defect, we already have a link state.]

(2) if both nodes  $j'$  and  $(j + 1)'$  are connected to arcs and the arcs intersect, use

(2.5) in [If both nodes  $j'$  and  $(j + 1)'$  are connected to arcs and the arcs do not intersect, we already have a link state.]

(3) if both nodes  $j'$  and  $(j + 1)'$  are connected to defects, then the other ends of the defects are attached to boxes of a standard Young tableau  $\mathbf{t}'$  that contain numbers  $k'$  and  $(k + 1)'$ ; then set

$$g_{j'} \left| \ell' \xrightarrow{\chi} \ell, \mathbf{t}', \mathbf{t} \right\rangle = \left| \ell' \xrightarrow{\chi} \ell, g_{k'} \cdot \mathbf{t}', \mathbf{t} \right\rangle,$$

where in the right-hand side the action of  $g_{k'}$  is that on a Specht module of  $\mathcal{H}_m(\alpha, \beta)$  (see **B.3.**)

- Acting with  $h_i$ :** (1) if node  $i$  is connected to a defect and node  $i + 1$  is connected to an arc, use (2.1). [If node  $i$  is connected to an arc and node  $i + 1$  is connected to a defect, we already have a link state.]
- (2) if both nodes  $i$  and  $i + 1$  are connected to arcs and the arcs intersect, use (2.1). [If both nodes  $i$  and  $i + 1$  are connected to arcs and the arcs do not intersect, we already have a link state.]
- (3) if both nodes  $i$  and  $i + 1$  are connected to defects, then the other ends of the defects are attached to boxes containing numbers  $k$  and  $k + 1$  of a standard Young tableau  $\mathbf{t}$ ; then set

$$h_i \left| \ell' \xrightarrow{\mathcal{X}} \ell, \mathbf{t}', \mathbf{t} \right\rangle = \left| \ell' \xrightarrow{\mathcal{X}} \ell, \mathbf{t}', h_k \cdot \mathbf{t} \right\rangle,$$

where in the right-hand side  $h_k$  acts on a Specht module of  $\mathcal{H}_n(\alpha, \beta)$  (see **B.3.**)

**4.1.3. Examples.** We act with the  $q\mathcal{WB}_{5,6}$  generators (or with their inverse, depending on which gives simpler expressions) on the link state  $\mathcal{X} = \left| (2' \mapsto 6, 5' \mapsto 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle$  shown in Fig. 3:

$$g_4^{-1} \mathcal{X} = \left| (2' \rightarrow 6, 4' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle$$

(for  $g_4$ , the first case of “Acting with  $g_j$ ” should be used here; equivalently,  $g_4^{-1}$  “disentangles” the configuration),

$$g_3^{-1} \mathcal{X} = \left| (2' \rightarrow 6, 5' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 2' \\ \hline 3' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle$$

(the action propagates to the Young tableau),

$$g_2 \mathcal{X} = \left| (3' \rightarrow 6, 5' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle$$

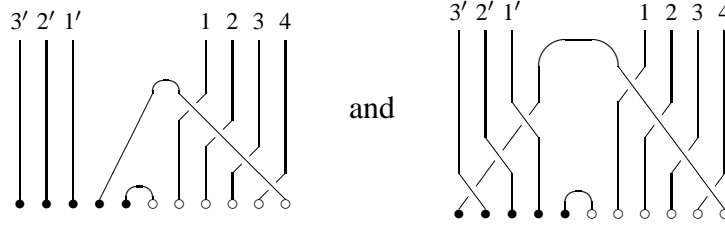
(a link state is obtained directly),

$$g_1^{-1} \mathcal{X} = \left| (1' \rightarrow 6, 5' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle$$

(the same pattern as for  $g_4$ ), and

$$\begin{aligned} \mathcal{E} \mathcal{X} &= \frac{\alpha^2 \beta^3}{\kappa} \left| (1' \rightarrow 1, 2' \rightarrow 6), \begin{array}{|c|c|} \hline 1' & 2' \\ \hline 3' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle - \frac{\alpha^2 \beta^2}{\kappa} \left| (1' \rightarrow 1, 2' \rightarrow 6), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle \\ &+ \frac{\alpha + \beta}{\kappa} \left| (1' \rightarrow 1, 5' \rightarrow 6), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle, \end{aligned}$$

where it is worth visualizing the two tangle configurations involved here:



Next,

$$\begin{aligned}
 h_1 \mathcal{X} &= \left| (2' \rightarrow 6, 5' \rightarrow 2), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle, \\
 h_2 \mathcal{X} &= \beta \left| (2' \rightarrow 6, 5' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle - \beta^2 \left| (2' \rightarrow 6, 5' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right\rangle, \\
 h_3^{-1} \mathcal{X} &= \left| (2' \rightarrow 6, 5' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right\rangle, \\
 h_4 \mathcal{X} &= \beta \left| (2' \rightarrow 6, 5' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle - \beta^2 \left| (2' \rightarrow 6, 5' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right\rangle, \\
 h_5^{-1} \mathcal{X} &= \left| (2' \rightarrow 5, 5' \rightarrow 1), \begin{array}{|c|c|} \hline 1' & 3' \\ \hline 2' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\rangle.
 \end{aligned}$$

**4.2.** The following lemma is an application of the link-state construction.

**4.2.1. Lemma.** *On the Specht module  $\mathcal{S}^{m,n,\lambda',\lambda}$ , the eigenvalue of the Casimir element  $C_{m,n}(1/(\alpha\beta))$  (see 2.5.1) is*

$$c_{m,n} = -(-\alpha\beta)^{m-n} \left( \frac{Z_1(\lambda')}{\theta} + Z_2(\lambda) \right),$$

where

$$Z_1(\lambda) = \sum_{\square \in \lambda} \left( -\frac{\beta}{\alpha} \right)^{\Gamma(\square)}, \quad Z_2(\lambda) = \sum_{\square \in \lambda} \left( -\frac{\beta}{\alpha} \right)^{-\Gamma(\square)}$$

are the diagram contents, whose definitions differ by a minus sign in the exponent;  $\Gamma(\square)$  is defined in A.3.

Similarly, the eigenvalue of  $\tilde{C}_{m,n}(1/(\alpha\beta))$  (see 2.5.3) is

$$\tilde{c}_{m,n} = -(-\alpha\beta)^{n-m} \left( \frac{Z_1(\lambda)}{\theta} + Z_2(\lambda') \right).$$

This lemma implies a necessary condition for Specht modules to belong to the same linkage class in the characteristic degenerate cases.

**4.2.2. Lemma.** *Let  $\theta = -\left(-\frac{\beta}{\alpha}\right)^r$  with  $r \in \mathbb{Z}$ . For a chosen  $j$  such that  $1 \leq j \leq f = m - |\lambda'|$ , let  $\Lambda'$  and  $\Lambda$  be Young diagrams obtained by adding  $j$  boxes to the respective*

diagrams  $\lambda'$  and  $\lambda$ . Then a necessary condition for  $\mathcal{S}^{m,n,\lambda',\lambda}$  and  $\mathcal{S}^{m,n,\Lambda',\Lambda}$  to be in the same linkage class is that

$$\sum_{i=1}^j \left(-\frac{\beta}{\alpha}\right)^{\Gamma(\square'_i)-r} = \sum_{i=1}^j \left(-\frac{\beta}{\alpha}\right)^{-\Gamma(\square_i)},$$

where  $\square'_i$  and  $\square_i$  are respectively the boxes of  $\Lambda'/\lambda'$  and  $\Lambda/\lambda$ .

## 5. SEMINORMAL REPRESENTATIONS AND THE SPECTRUM OF JUCYS–MURPHY ELEMENTS

In this section, we diagonalize the Jucys–Murphy elements of  $\text{qw}\mathcal{B}_{m,n}(\alpha, \beta, \kappa, \theta)$  at generic values of the parameters. For this, we construct seminormal  $\text{qw}\mathcal{B}_{m,n}$  representations.

As in the preceding section, we temporarily fix  $m$  and  $n$ , and two Young diagrams  $\lambda'$  and  $\lambda$  such that  $m - |\lambda'| = n - |\lambda| =: f \geq 0$  (either diagram, or both, can be empty).

The seminormal representation  $\mathcal{L}^{m,n,\lambda',\lambda}$  has a seminormal basis of *standard triples*. These are introduced in 5.1; for each standard triple, we define its weight in 5.2, which eventually turns out to be the set of eigenvalues of the Jucys–Murphy elements; the crucial part of the construction is the  $\text{qw}\mathcal{B}_{m,n}$  action on standard triples, which is defined in 5.3. The proof of Theorem 5.4, which asserts the  $\text{qw}\mathcal{B}_{m,n}$  action, is given in 5.5. In 5.6, we define an invariant scalar product on seminormal representations. In 5.7, we finally prove the diagonalization of Jucys–Murphy elements.

### 5.1. Standard triples and mobile elements.

**5.1.1. Standard triples.** Let  $\tilde{\lambda}$  be a Young diagram obtained by adding  $f$  boxes to  $\lambda$ . A standard triple  $(B, S, W)$  (essentially as in [43], whence the term is borrowed) is a filling of  $(\lambda', \tilde{\lambda}/\lambda, \tilde{\lambda})$  in accordance with the following rules.

- (1) The disjoint union of  $\lambda'$  and  $\tilde{\lambda}/\lambda$  is filled with  $1', \dots, m'$  such that  $\lambda'$  is made into a standard tableau  $B$ , and  $\tilde{\lambda}/\lambda$  into an antistandard skew-shape tableau  $S$  (with the entries strictly decreasing along rows and along columns).
- (2)  $\tilde{\lambda}$  is filled with  $1, \dots, n$  into a standard tableau  $W$ .

We let  $\mathbf{T} = \mathbf{T}_{m,n}(\lambda', \lambda)$  be the set of all standard triples  $(B, S, W)$  constructed this way. We repeat that, for a fixed  $\lambda$ , the shapes of  $S$  and  $W$  are related as  $\overline{S} = \overline{W}/\lambda$ , which we also express as  $\overline{W} = \lambda \sqcup \overline{S}$ .

We refer to the three tableaux in a standard triple  $\mathbf{t} = (B, S, W)$  as the *black*, *skew*, and *white* ones for the obvious reason of shape combined with the “color scheme” used throughout this paper.

**5.1.2. Example.** If  $m = 6, n = 6, \lambda' = \begin{smallmatrix} \square \\ \square \end{smallmatrix}$ , and  $\lambda = \square\square$ , with  $f = 4$ , a possible standard triple is

$$(B, S, W) = \left( \begin{array}{|c|} \hline 1' \\ \hline 4' \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & 5' \\ \hline 6' & 3' & 2' \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \right).$$

Another example is

$$(B, S, W) = \left( \begin{array}{|c|} \hline 2' \\ \hline 5' \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & 4' \\ \hline 6' & 3' & \\ \hline 1' & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline 6 & & \\ \hline \end{array} \right).$$

**5.1.3. Mobile elements.** Given a standard triple  $(B, S, W)$ , we write  $S \diamond W$  for  $S$  superimposed on  $W$ . If  $1'$  from  $S$  and the largest number in  $W$  (which is  $n$ ) then occur in the same box, we say that this box is a *mobile element* (otherwise  $S \diamond W$  contains no mobile element).

In the first example in **5.1.2**, there is no mobile element. In the second example, this is the box containing  $6_{1'}$  in

$$S \diamond W = \begin{array}{|c|c|c|} \hline 1 & 3 & 4_{4'} \\ \hline 2_{6'} & 5_{3'} & \\ \hline 6_{1'} & & \\ \hline \end{array}$$

The fundamental property of mobile elements is that they can travel.<sup>12</sup> By moving a mobile element we mean detaching it from the rest of  $S \diamond W$  and reattaching to the remainder in a new position,  $(S, W) \rightarrow (S_1, W_1)$ ; the shapes are here related as  $\overline{W_1} = \lambda \sqcup \overline{S_1}$ , and the resulting  $W_1$  is a standard tableau and  $S_1$  is a skew antistandard tableau built on a skew shape.

The *orbit* of a mobile element is the set of all positions into which it can be moved (including the original position). The orbit of  $6_{1'}$  in the above example is shown with stars:

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4_{4'} \\ \hline 2_{6'} & 5_{3'} & \\ \hline \end{array} \begin{array}{c} \star \\ \star \\ \star \end{array}$$

We can speak of the orbit in terms of just the  $S$  or just the  $W$  part of the triple, because for any  $(S_1, W_1)$  from the orbit of  $(S, W)$ ,  $W_1$  is uniquely reconstructed from  $S_1$ , and vice versa. In many cases in what follows, the existence of a mobile element is assumed; it then suffices to specify how the box containing  $1'$  travels, and we therefore write  $\text{Orb}_{1'}(\lambda \sqcup S)$  for the orbit.

<sup>12</sup>A crucial property in a different context [44].

**5.2. Weights.** For a standard triple  $(B, S, W)$ , we define its weight  $\text{Wt}(\mathbf{t}) \in \mathbb{C}^{m+n-1}$  as follows. The first  $n-1$  components of this  $(m+n-1)$ -component vector are the weight of  $W$  viewed as a basis element in a seminormal representation of the Hecke algebra  $\mathcal{H}_n(\alpha, \beta)$  (see **B.4.1**):

$$\text{Wt}(B, S, W)_i = \text{wt}(W)_i, \quad 1 \leq i \leq n-1.$$

The remaining components are determined by how  $(B, S)$  is filled with the numbers  $1', \dots, m'$ :

$$\text{Wt}(B, S, W)_{n-1+i'} = \begin{cases} \left(-\frac{\beta}{\alpha}\right)^{\Gamma(B[i'])}, & i' \in B, \\ -\theta \left(-\frac{\beta}{\alpha}\right)^{-\Gamma(S[i'])}, & i' \in S, \end{cases} \quad 1 \leq i' \leq m,$$

where, to recall,  $\mathbf{t}[a]$  is the position (coordinates on the plane) of a number  $a$  in a Young tableau  $\mathbf{t}$  (see **A.2**) and  $\Gamma(\cdot)$  is defined in **A.3**.

**5.2.1. Example.** To continue with the examples in **5.1.2**, the corresponding weights are

$$\left(-\frac{\alpha}{\beta}, -\frac{\beta}{\alpha}, \frac{\alpha^2}{\beta^2}, 1, -\frac{\alpha}{\beta}; 1, \frac{\beta\theta}{\alpha}, -\theta, -\frac{\beta}{\alpha}, -\frac{\beta^2\theta}{\alpha^2}, \frac{\alpha\theta}{\beta}\right)$$

and

$$\left(-\frac{\beta}{\alpha}, -\frac{\alpha}{\beta}, \frac{\alpha^2}{\beta^2}, 1, \frac{\beta^2}{\alpha^2}; -\frac{\alpha^2\theta}{\beta^2}, 1, -\theta, -\frac{\beta^2\theta}{\alpha^2}, -\frac{\beta}{\alpha}, \frac{\alpha\theta}{\beta}\right),$$

with the first  $n-1$  components (which are  $\text{wt}(W)$ ) separated by a semicolon for clarity.

**5.3. The  $\text{qw}\mathcal{B}_{m,n}$  action on a seminormal basis.** The standard triples  $\mathbf{T}_{m,n}(\lambda', \lambda)$  are a basis in the seminormal representation  $\mathcal{L}^{m,n,\lambda',\lambda}$ . The action of  $\text{qw}\mathcal{B}_{m,n}$  generators on  $\mathbf{T}_{m,n}(\lambda', \lambda)$  is defined in **5.3.1**, **5.3.2**, and **5.3.3** below.

For  $(B, S, W) \in \mathbf{T}_{m,n}(\lambda', \lambda)$ , we let  $(\mu_1, \dots, \mu_{m+n-1}) = \text{Wt}(B, S, W)$ .

**5.3.1.** For  $(B, S, W) \in \mathbf{T}_{m,n}(\lambda', \lambda)$ , we define the action of  $h_i$ ,  $1 \leq i \leq n-1$ , in accordance with (B.6):

$$h_i.(B, S, W) = (B, S, h_i.W)$$

(with the obvious linearity assumed here and in what follows), where

$$(5.1) \quad h_{n-i}.W = \begin{cases} \alpha W, & i \text{ and } i+1 \text{ are in the same row,} \\ \beta W, & i \text{ and } i+1 \text{ are in the same column,} \\ \frac{\alpha+\beta}{1-\frac{\mu_{i-1}}{\mu_i}} W + \eta\left(\frac{\mu_{i-1}}{\mu_i}\right) W_{(i,i+1)} & \text{otherwise.} \end{cases}$$

**5.3.2.** The action of  $g_j$  on a standard triple  $(B, S, W)$  is

$$g_j.(B, S, W) = (g_j.(B, S), W), \quad 1 \leq j \leq m-1,$$

where



$$(5.2) \quad g_{j \cdot}(B, S) = \begin{cases} \alpha \cdot (B, S), & j \text{ and } j+1 \text{ are in the same row of } B \text{ or } S, \\ \beta \cdot (B, S), & j \text{ and } j+1 \text{ are in the same column of } B \text{ or } S, \\ \frac{\alpha + \beta}{1 - \frac{\mu_{n+j-1}}{\mu_{n+j}}} \cdot (B, S) + \zeta\left(\frac{\mu_{n+j-1}}{\mu_{n+j}}\right)(B, S)_{(j, j+1)}, & \text{otherwise,} \end{cases}$$

where  $(B, S)_{(j, j+1)}$  is obtained from  $(B, S)$  by the transposition of  $j$  and  $j+1$ . The weight components in the third line depend on  $B$  and  $S$ , but are independent of  $W$ . Similarly to (B.7), the function  $\zeta$  involved there is such that

$$(5.3) \quad \zeta(x)\zeta\left(\frac{1}{x}\right) = -\alpha\beta F(x)$$

(see (B.8) for  $F(x)$ ). We in addition require that

$$(5.4) \quad \frac{\eta(x)\zeta(x)\eta(y)\zeta(y)}{\eta(xy)\zeta(xy)} = -\alpha\beta \frac{F(x)F(y)}{F(xy)},$$

a condition needed for consistency, as we see in what follows.

A “totally symmetric” choice satisfying Eqs. (5.3), (B.7), and (5.4) is

$$(5.5) \quad \zeta(x) = \eta(x) = \sqrt{-\alpha\beta F(x)}.$$

Some formulas in what follows are essentially simplified with this choice, but keeping general  $\eta$  and  $\zeta$  subject to Eqs. (B.7), (5.3), and (5.4) is quite useful in the proofs, because it “explains” the occurrence of different terms.<sup>13</sup>

**5.3.3.** The action of  $\mathcal{E}$  on a standard triple  $(B, S, W)$  involves the mobile element (see **5.1.3**). For brevity, we write  $S_1 \in \text{Orb}_{1'}(\lambda \sqcup S)$  instead of  $(S_1, W_1) \in \text{Orb}(S, W)$  for elements in the orbit of the mobile element, with the understanding that (for a given  $\lambda$ ) each  $S_1$  from the orbit uniquely defines an appropriate  $W_1$ .

For a Young diagram  $\lambda$ , we let  $\underline{\lambda}$  denote its corners (removable boxes):

$$\underline{\lambda} = \{\square \in \lambda \mid \square \text{ is removable}\}.$$

The definition naturally extends to Young tableaux. Also, if a (skew-shaped) tableau  $S$  contains  $1'$ , then we let  $S \boxminus 1'$  denote the result of removing the box containing  $1'$ .

Then

$$(5.6) \quad \mathcal{E} \cdot (B, S, W) = \begin{cases} 0, & \text{no mobile element in } S \diamond W, \\ \sum_{S_1 \in \text{Orb}_{1'}(\lambda \sqcup S)} c_{S, S_1} \cdot (B, S_1, W_1) & \text{otherwise,} \end{cases}$$

where

$$c_{S, S_1} = \frac{1}{\kappa(\alpha + \beta)} c_S^{(1)} c_{S_1}^{(2)},$$

<sup>13</sup>The general solution of (5.4) for  $\eta(x)\zeta(x)$  is  $\eta(x)\zeta(x) = -\alpha\beta x^Q F(x)$ .

We next illustrate the entire formula for the  $\mathcal{E}$  action. Taking  $m = 2$  and  $n = 4$ , we consider the 16-dimensional seminormal representation defined by the pair of Young

diagrams  $(\lambda', \lambda) = (\square, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$ . An instance of the  $\mathcal{E}$  action on a standard triple from  $\mathbf{T}_{2,4}(\lambda', \lambda)$  is

$$\begin{aligned} \mathcal{E} \cdot \left( \begin{smallmatrix} \square \\ 2' \end{smallmatrix}, \cdot \cdot \cdot \begin{smallmatrix} \square \\ 1' \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square \\ 1 & 3 & 4 \\ 2 & & \end{smallmatrix} \right) &= \frac{1+q^2\theta}{\kappa(\alpha+\beta)} \left( \frac{(1-q^{-3})(1-q^{-1})}{(1-q^{-4})(1-q^{-2})} \left( \begin{smallmatrix} \square \\ 2' \end{smallmatrix}, \cdot \cdot \cdot \begin{smallmatrix} \square \\ 1' \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ 1 & 3 \\ 2 & \\ & 4 \end{smallmatrix} \right) \right. \\ &+ \frac{(1-q)(1-q^3)}{(1-q^2)(1-q^4)} \left( \begin{smallmatrix} \square \\ 2' \end{smallmatrix}, \cdot \cdot \cdot \begin{smallmatrix} \square \\ 1' \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square \\ 1 & 3 & 4 \\ 2 & & \end{smallmatrix} \right) + \left. \frac{(1-q^{-1})(1-q)}{(1-q^{-2})(1-q^2)} \left( \begin{smallmatrix} \square \\ 2' \end{smallmatrix}, \cdot \cdot \cdot \begin{smallmatrix} \square \\ 1' \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ 1 & 3 \\ 2 & 4 \end{smallmatrix} \right) \right), \end{aligned}$$

where the dots are for  $\lambda$ , showing how the “skew shape,” which here consists of a single box, is to be completed to a suitable Young diagram.

**5.4. Theorem.** *The formulas in 5.3.1, 5.3.2, and 5.3.3 define a  $\text{qwB}_{m,n}$  representation.*

**5.5. Proof.** We need to prove the “genuinely  $\text{qwB}$ ” relations, i.e., those involving  $\mathcal{E}$ , Eqs. (2.13)–(2.16); the relations in the Hecke subalgebras are standard; that the  $g_j$  commute with  $h_i$  is also obvious.

We begin with relations (2.13) and (2.14), and first rewrite them in a suitable form showing that they depend only on a Young diagram and a chosen corner (actually, on the diagram obtained by removing that corner).

**5.5.1.** As regards the idempotent property (2.13), we calculate

$$\mathcal{E} \cdot \mathcal{E} \cdot (B, S, W) = \frac{1}{\kappa(\alpha+\beta)} \rho_{\lambda \sqcup S} \mathcal{E} \cdot (B, S, W),$$

where, directly from the definitions,

$$\rho_{\lambda \sqcup S} = \sum_{S_1 \in \text{Orb}_{1'}(\lambda \sqcup S)} c_{S_1}^{(2)} c_{S_1}^{(1)} = \sum_{S_1 \in \text{Orb}_{1'}(\lambda \sqcup S)} \frac{\prod_{\delta \in \overline{\lambda \sqcup S_1 \sqcup 1'}} (1 - q^{\Gamma(\delta, S_1[1'])})}{\prod_{\substack{S_2 \in \text{Orb}_{1'}(\lambda \sqcup S_1) \\ S_2 \neq S_1}} (1 - q^{\Gamma(S_2[1'], S_1[1'])})} \left( 1 + \frac{\theta}{q^{\Gamma(S_1[1'])}} \right).$$

Relation (2.13) is equivalent to

$$\rho_{\lambda \sqcup S} = \theta + 1,$$

which involves two identities—for the terms proportional to and free of  $\theta$ —which are equivalent to each other and which can be reformulated as follows. For a given Young diagram  $\Lambda$ , we let  $\overline{\Lambda}$  denote the (positions of) boxes addable to it. Then (2.13) holds for the action of  $\mathcal{E}$  defined in 5.3.3 if and only if

$$(5.8) \quad \sum_{\star \in \overline{\Lambda}} \frac{\prod_{\circ \in \Lambda} (1 - q^{\Gamma(\star, \circ)})}{\prod_{\star' \neq \star} (1 - q^{\Gamma(\star, \star')})} = 1$$

for any Young diagram  $\Lambda$ . Here,  $\circ$  ranges over corners of the diagram and  $\star$  ranges over the boxes addable to it.

We next reduce relations (2.14) to a similar identity. We select the relation  $\mathcal{E}g_1\mathcal{E} = \frac{1}{\kappa}\mathcal{E}$  for definiteness. It is nontrivial in the seminormal representation only when applied to a standard triple containing a mobile element; but the nondiagonal part of the action of  $g_1$  destroys this mobile element ( $1'$  is no longer in the same box with  $n$  in  $S\Diamond W$ ); hence, only the diagonal part of the  $g_1$  action in the third line in (5.2) makes a contribution. Then the coefficient in front of the first term in the formula for the  $g_1$  action,

$$\frac{\alpha + \beta}{1 - \mu_n \mu_{n+1}^{-1}} = \frac{\alpha + \beta}{1 - q^{\Gamma(S[2'], S[1'])}},$$

involves the hook distance between the boxes containing  $1'$  and  $2'$ . But  $2'$  can stand only in a corner of  $\lambda \sqcup S \sqcup 1'$ ; thus,  $\mathcal{E}g_1\mathcal{E} \cdot (B, S, W)$  turns out to be proportional to  $\mathcal{E} \cdot (B, S, W)$ , with the coefficient similar to  $\rho_{\lambda \sqcup S}$  above, but with one of the factors corresponding to  $\lambda \sqcup S_1 \sqcup 1'$  in the numerator missing. It thus follows that relations (2.14) (each of them, as is easy to see) hold in the seminormal representation if and only if the identity

$$(5.9) \quad \sum_{\star \in \Gamma\Lambda} \frac{\prod_{\substack{\circ \in \underline{\Lambda}, \\ \circ \neq \star}} (1 - q^{\Gamma(\star, \circ)})}{\prod_{\star' \neq \star} (1 - q^{\Gamma(\star, \star')})} = 1,$$

holds for any Young diagram  $\Lambda$  with a fixed corner  $\star \in \underline{\Lambda}$ .

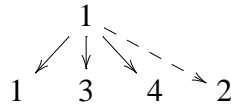
Identities (5.8) and (5.9) are particular cases of two-variate identities established for any Young diagram in Appendix C. This proves (2.13) and (2.14) for the  $qwB$  action on standard triples.

**5.5.2.** It remains to establish quintic identities (2.16) for the  $qwB$  action on standard triples. For this, we calculate how the element  $\Omega = \mathcal{E}g_1h_1^{-1}\mathcal{E}$  involved in these identities acts on the seminormal basis. First of all,  $\mathcal{E} \cdot (B, S, W)$  is nonzero if and only if a corner of  $S\Diamond W$  is a mobile element  $\underline{n_{1'}}$ . Next,  $h_1^{-1}$  acts depending on the relative position of  $n-1$  and  $n$  in  $W$  and  $g_1$  acts depending on the relative position of  $2'$  and  $1'$  in  $S$  (it is obvious that if  $2' \in B$ , then the action of  $\Omega$  gives zero). We list the possible configurations of the relevant corners of  $S\Diamond W$ .

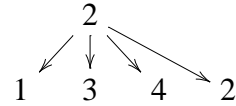
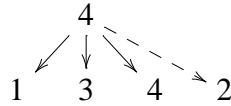
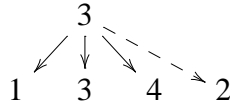
- (1) The superimposed tableau  $S\Diamond W$  has  $\underline{n_{1'}}$ ,  $\underline{(n-1)_{2'}}$ ,  $\underline{?_{2'}}$  as (some of) its corners (where the unprimed question mark is an integer less than  $n-1$  and the primed one is an integer greater than 2);
- (2) boxes near a corner of  $S\Diamond W$  are configured as  $\begin{array}{|c|} \hline \underline{?_{2'}} \\ \hline \underline{(n-1)_{2'}} \quad \underline{n_{1'}} \\ \hline \end{array}$ ; or the configuration with  $(n-1)_{2'}$  and  $?_{2'}$  transposed is realized.

- (3)  $S \diamond W$  has a row ending with  $\underline{(n-1)_{2'} \mid n_{1'}}$  and a corner  $\underline{?_{2'}}$ ; or a “transposed” case is realized: a column ends with  $n_{1'}$  just below  $(n-1)_{2'}$  and another corner contains  $2'$ ;
- (4)  $S \diamond W$  has a row ending with  $\underline{?_{2'} \mid n_{1'}}$  and a corner  $\underline{(n-1)_{2'}}$ ; or the case where the row is transposed into a column is realized;
- (5)  $S \diamond W$  has corners  $\underline{n_{1'}}$ ,  $\underline{(n-1)_{2'}}$ ;
- (6)  $S \diamond W$  has a row ending with  $\underline{(n-1)_{2'} \mid n_{1'}}$ ; or a transposed case is realized;

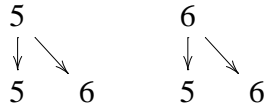
For  $i = 1, \dots, 6$ , we say that a tableau is in class  $i$  if its configuration of relevant corners is described in item  $i$  of the above list. We see immediately that the action of  $\mathcal{E}$  on a tableau in class 1 produces tableaux from classes 1, 3, and 4 and, if the original tableau has corners arranged as  $\frac{\underline{(n-1)_{2'}}}{\underline{?_{2'}}}$ , also a tableau from class 2. We write this as



We also have



and, similarly,



We next see how  $h_1^{-1} g_1$  acts on the tableaux from each class. We recall that when the relevant action is in accordance with the third line in (5.2), we have

$$(5.10) \quad g_1 \cdot (B, S, W) = \frac{\alpha + \beta}{1 - \frac{\mu_n}{\mu_{n+1}}} (B, S, W) + \zeta \left( \frac{\mu_n}{\mu_{n+1}} \right) ((B, S)_{(1', 2')}, W)$$

and, similarly (from the third line in (5.1)),

$$(5.11) \quad h_1^{-1} \cdot (B, S, W) = \frac{\alpha + \beta}{\alpha \beta (1 - \frac{\mu_{n-1}}{\mu_{n-2}})} (B, S, W) + \frac{1}{\alpha \beta} \eta \left( \frac{\mu_{n-2}}{\mu_{n-1}} \right) (B, S, W_{(n-1, n)}).$$

where  $(\mu_1 \dots, \mu_{m+n-1}) = \text{Wt}(B, S, W)$  as usual. We use these formulas in the case where both  $1'$  and  $2'$  are in  $S$ ; then not only the ratio  $\frac{\mu_{n-1}}{\mu_{n-2}}$  but also  $\frac{\mu_n}{\mu_{n+1}}$  is expressible in terms of hook distance:

$$\frac{\mu_{n-1}}{\mu_{n-2}} = q^{\Gamma(n, n-1)}, \quad \frac{\mu_n}{\mu_{n+1}} = q^{\Gamma(2', 1')}.$$

where we write  $\Gamma(n, n-1) = \Gamma(W[n], W[n-1])$  and  $\Gamma(2', 1') = \Gamma(S[2'], S[1'])$  for brevity.

The following facts are easy to verify directly from the definitions.

- (1) In class 1, the action of  $h_1^{-1}g_1$ , in addition to the original tableau, produces tableaux with  $(2', 1')$  or/and  $(n-1, n)$  transposed; none of these has a mobile element, and hence all “new” triples vanish under the subsequent action of  $\mathcal{E}$ . Thus, effectively (“inside  $\Omega$ ”),  $h_1^{-1}g_1$  acts by the corresponding eigenvalue.
- (2) In class 2, both  $h_1^{-1}$  and  $g_1$  act literally by eigenvalues.
- (3) In class 3,  $h_1^{-1}$  acts by an eigenvalue, and hence the configurations produced additionally by the action of  $h_1^{-1}g_1$  are those with  $\underline{(n-1)_{2'} \mid n_{2'}}, \underline{?_{1'}}$ , which are annihilated by  $\mathcal{E}$  (and similarly in the transposed case, which we do not mention explicitly any more). Effectively, therefore,  $h_1^{-1}g_1$  acts again by an eigenvalue.
- (4) In class 4,  $g_1$  acts by an eigenvalue, and the configurations produced additionally by the action of  $h_1^{-1}g_1$  are those with  $\underline{?_{2'} \mid (n-1)_{1'}}, \underline{n_{2'}}$ , and are also annihilated by  $\mathcal{E}$ ; hence,  $h_1^{-1}g_1$  effectively acts by an eigenvalue.
- (5) In class 5, the “new” corners resulting from the action of  $h_1^{-1}g_1$  are  $\underline{(n-1)_{1'}}, \underline{n_{2'}}$ , and such tableaux are annihilated by  $\mathcal{E}$ ; but the “old” corners  $\underline{(n-1)_{2'}}$  and  $\underline{n_{1'}}$  can now occur in both the original and transposed positions, and we readily find

$$h_1^{-1}g_1 \Big|_5 : (B, S, W) \mapsto \frac{(1-q)(1-q^{-1})}{(1-q^{\Gamma(2', 1')})(1-q^{\Gamma(n, n-1)})} (B, S, W) - \frac{1}{\alpha\beta} \eta(q^{\Gamma(2', 1')}) \zeta(q^{-\Gamma(n, n-1)}) (B, S_{(1', 2')}, W_{(n, n-1)}).$$

- (6) In class 6, both  $h_1^{-1}$  and  $g_1$  act literally by eigenvalues, and

$$h_1^{-1}g_1 \Big|_6 : (B, S, W) \mapsto (B, S, W).$$

**5.5.3. Lemma.** *We have*

$$\Omega \Big|_{1,3,4,2} = 0.$$

Moreover,  $\Omega.(B, S, W) \neq 0$  if and only if  $S \diamond W$  has corners containing  $n_{1'}$  and  $(n-1)_{2'}$ .

**5.5.4. Proof of 5.5.3.** The key observation is that the eigenvalues in classes 1, 3, 4, and 2 are expressed the same:

$$(5.12) \quad h_1^{-1}g_1 \Big|_{1,3,4,2} : (B, S, W) \mapsto \omega_{(S, W)}(B, S, W),$$

$$\omega_{(S, W)} = \frac{(1-q)(1-q^{-1})}{(1-q^{\Gamma(2', 1')})(1-q^{\Gamma(n, n-1)})}.$$

(In class 2, this does reduce to the formula found directly,  $h_1^{-1}g_1 \Big|_2 : (B, S, W) \mapsto -q^{\Gamma(1', 2')}. (B, S, W)$ , because  $\Gamma(1', 2') = \Gamma(n-1, n) = \pm 1$  in that case.) We then calculate

$$\Omega.(B, S, W) = \sum_{S_1 \in \text{Orb}_{1'}(\lambda \sqcup S)} c_{S, S_1} \mathcal{E} h_1^{-1}g_1.(B, S_1, W_1)$$

$$= \sum_{S_1 \in \text{Orb}_{1'}(\lambda \sqcup S)} \sum_{S_2 \in \text{Orb}_{1'}(\lambda \sqcup S_1)} \omega_{(S_1, W_1)} c_{S, S_1} c_{S_1, S_2} (B, S_2, W_2)$$

(where, of course,  $\text{Orb}_{1'}(\lambda \sqcup S_1) = \text{Orb}_{1'}(\lambda \sqcup S)$ ) and recall the factored structure of the coefficient in (5.7). The right-hand side of the last formula vanishes because

$$\sum_{S_1 \in \text{Orb}_{1'}(\lambda \sqcup S)} \omega_{(S_1, W_1)} c_{S_1}^{(2)} c_{S_1}^{(1)} = 0,$$

which is yet another identity from the class established in Appendix C; compared with (5.8), *two* factors are missing in each denominator—those canceled by the two factors in (5.12) involving  $\Gamma(2', 1')$  and  $\Gamma(n, n-1)$ , which, up to a sign, are the distances from the mobile element to two corners of  $\lambda \sqcup S \sqcup 1'$ .

**5.5.5.** With the vanishing of  $\Omega$  on classes 1, 3, 4, and 2 established in the lemma, it remains to calculate  $\Omega.(B, S, W)$  in the cases where  $S \diamond W$  is in classes 5 and 6, i.e., contains boxes (corners)  $n_{1'}$  and  $(n-1)_{2'}$ .<sup>14</sup> We note that in terms of the weight  $(\mu_1, \dots, \mu_{m+n-1}) = \text{Wt}(B, S, W)$ , this condition is equivalently stated as

$$\begin{cases} \mu_{n-1} \mu_n = -\theta, \\ \mu_{n-2} \mu_{n+1} = -\theta. \end{cases}$$

In particular,

$$\frac{\mu_{n-1}}{\mu_{n-2}} = \frac{\mu_{n+1}}{\mu_n},$$

which we extensively use in what follows.

For a standard triple  $(B, S, W)$  in class 5 or 6, we now see that  $\Omega.(B, S, W)$  is a sum over *the 1'2'-orbit of  $(B, S, W)$* —all tableaux obtained by detaching the boxes containing  $1'$  and  $2'$  from  $\lambda \sqcup S$  and reattaching them so as to obtain an antistandard tableau:

$$(5.13) \quad \Omega.(B, S, W) = \sum_{S_1 \in \text{Orb}_{1'2'}(\lambda \sqcup S)} \frac{1}{\kappa^2} z_{S, S_1} (B, S_1, W_1),$$

with the coefficients  $z_{S, S_1}$  to be found below; here and hereafter, we let the  $1'2'$ -orbit be denoted by  $\text{Orb}_{1'2'}(\lambda \sqcup S)$ , once again with the understanding that each  $W_1$  in the sum is uniquely determined by  $\lambda \sqcup S_1$  just because the position of  $n$  in  $W_1$  coincides with the position of  $1'$  in  $S_1$  and the position of  $n-1$  coincides with the position of  $2'$ .

We need more notation to proceed.

**5.5.6. Notation.** We continue operating in terms of not the superimposed tableaux  $S \diamond W$  (now assumed to contain the boxes  $n_{1'}$  and  $(n-1)_{2'}$ ) but the antistandard tableaux  $S$  such that  $1', 2' \in S$ ; it is understood that every rearrangement of  $S$  is followed by the corresponding rearrangement of  $W$ , such that  $n$  travels together with  $1'$  and  $(n-1)$  together with  $2'$ .

<sup>14</sup>This is only possible in  $\mathcal{L}^{m, n, \lambda', \lambda}$  with  $m - |\lambda'| \geq 2$ . In representations with  $m - |\lambda'| < 2$ ,  $\Omega$  acts by zero.

- (1) We let  $\lambda \sqcup S \sqsupset 1'2'$  denote corners of  $\lambda \sqcup S \sqsupset 1' \sqsupset 2'$ , the tableau with boxes  $1'$  and  $2'$  removed.
- (2) We also let  $\lambda \sqcup S \sqsupset 1' \big|_{\neq 2'}$  denote the corners of  $\lambda \sqcup S \sqsupset 1'$  except the box containing  $2'$ . In two examples below, the elements of  $\lambda \sqcup S \sqsupset 1' \big|_{\neq 2'}$  are shown in color:

		2'	1'

			2'
		1'	

- (3) We define  $S \sqsupset 2'$ , a skew antistandard tableau with the box  $2'$  removed. The definition is obvious if  $S$  contains two corners  $1'$  and  $2'$ ; otherwise — if  $S$  contains  $\begin{smallmatrix} 2' & 1' \end{smallmatrix}$  or  $\begin{smallmatrix} 2' \\ 1' \end{smallmatrix}$  — we let  $S \sqsupset 2'$  be the skew tableau with the box  $2'$  removed *and*  $1'$  taking its place. For example,

$$(5.14) \quad \lambda \sqcup S = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & 2' & 1' \\ \hline \end{array} \implies \lambda \sqcup S \sqsupset 2' = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & 1' \\ \hline \end{array}.$$

- (4) We let  $\text{Orb}_{2'}(\lambda \sqcup S \sqsupset 2')$  denote all possible ways to attach  $2'$  to  $\lambda \sqcup S \sqsupset 2'$  so as to obtain an antistandard tableau.

For  $\lambda \sqcup S$  in (5.14), for example,  $\text{Orb}_{2'}(\lambda \sqcup S \sqsupset 2')$  consists of two elements marked with \*:

$$\begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & 1' & & \\ \hline \end{array} * \quad \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & 1' & & \\ \hline \end{array} *$$

We also illustrate the definition of  $\text{Orb}_{1'2'}(\lambda \sqcup S)$  given just above (5.13). For

$$\lambda \sqcup S = \begin{array}{|c|c|c|} \hline \cdot & \cdot & 2' \\ \hline 6' & 5' & \\ \hline 1' & & \\ \hline \end{array},$$

its  $1'2'$ -orbit is this tableau itself together with five more:

$$\begin{array}{|c|c|c|} \hline \cdot & \cdot & 1' \\ \hline 6' & 5' & \\ \hline 2' & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & 2' & 1' \\ \hline 6' & 5' & & \\ \hline & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & 2' & \\ \hline 6' & 5' & 1' & \\ \hline & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline \cdot & \cdot & \\ \hline 6' & 5' & \\ \hline 2' & 1' & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline \cdot & \cdot & \\ \hline 6' & 5' & \\ \hline 2' & & 1' \\ \hline \end{array}.$$

**5.5.7.** After these preparations, tracking the occurrence of the different factors in (5.13) gives the following lemma, where we continue using  $q = -\frac{\beta}{\alpha}$  for brevity.

**5.5.8. Lemma.** *We have*

$$z_{S,T} = \begin{cases} -\frac{X_S Y_T}{\alpha \beta}, & S[2'] = T[2'], \\ \frac{X_S Y_T}{(\alpha \beta)^2} \frac{\eta(q^{\Gamma(S[2'], T[2'])}) \zeta(q^{\Gamma(S[2'], T[2'])})}{F(q^{\Gamma(S[2'], T[2'])})} & \text{otherwise,} \end{cases}$$



where  $F$  is defined in (B.8) and the coefficients are

$$X_S = \left(1 + \frac{\theta}{q^{\Gamma(S[1'])}}\right) \left(1 + \frac{\theta}{q^{\Gamma(S[2'])}}\right),$$

$$Y_S = \frac{C_S q^{\Gamma(S[2'], S[1'])}}{(1-q)(1-q^{-1})} \frac{\prod_{\delta \in \lambda \sqcup S \boxminus 1'} (1 - q^{\Gamma(\delta, S[1'])})}{\prod_{\substack{S_1 \in \text{Orb}_{1'}(\lambda \sqcup S) \\ S_1 \neq S}} (1 - q^{\Gamma(S_1[1'], S[1'])})} \frac{\prod_{\delta \in \lambda \sqcup S \boxminus 1' 2'} (1 - q^{\Gamma(\delta, S[2'])})}{\prod_{\substack{S_1 \in \text{Orb}_{2'}(\lambda \sqcup S \boxminus 1' 2') \\ S_1 \neq S}} (1 - q^{\Gamma(S_1[2'], S[2'])})},$$

and

$$C_S = \begin{cases} 1 - q^{-\Gamma(S[2'], S[1'])}, & S \text{ contains } \begin{smallmatrix} 2' & 1' \\ 1' \end{smallmatrix} \text{ or } \begin{smallmatrix} 2' \\ 1' \end{smallmatrix}, \\ 1, & \text{otherwise.} \end{cases}$$

We write  $X_S$ ,  $Y_S$ , etc., instead of the more rigorous  $X_{\lambda \sqcup S}$ ,  $Y_{\lambda \sqcup S}$ , etc.

**5.5.9. Examples.** For  $\lambda \sqcup S$  in (5.14),

$$Y_S = -\frac{q^2(1-q^{-1})}{(1-q)^2} \frac{1-q^{-2}}{(1-q^{-3})(1-q^3)} \frac{(1-q^{-3})(1-q)}{(1-q^{-4})(1-q^2)}.$$

Another example is

$$\lambda \sqcup S = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & 2' & \\ \hline & & & \\ \hline 1' & & & \\ \hline \end{array} \implies Y_S = \frac{-q^{-3}}{(1-q)^2} \frac{(1-q^{-6})(1-q^{-1})}{(1-q^{-7})(1-q^{-5})(1-q^{-2})} \frac{(1-q^{-2})(1-q)(1-q^3)}{(1-q^{-3})(1-q^2)}.$$

**5.5.10. The end of the proof of 5.4.** With all the instances where  $\Omega.(B, S, W)$  is nonzero calculated in 5.5.7 and 5.5.8, we can finally calculate  $(h_1 - g_1)\Omega$  on any vector of the seminormal basis. The diagonal parts of the action of  $g_1$  in (5.10) and of  $h_1$  in

$$h_1.(B, S, W) = \frac{\alpha + \beta}{1 - \frac{\mu_{n-2}}{\mu_{n-1}}} (B, S, W) + \eta\left(\frac{\mu_{n-2}}{\mu_{n-1}}\right) (B, S, W_{(n-1, n)})$$

cancel when  $g_1 - h_1$  is applied to each term in (5.13); in particular, all tableaux containing  $\begin{smallmatrix} 2' \\ 1' \end{smallmatrix}$  or  $\begin{smallmatrix} 2' & 1' \\ 1' \end{smallmatrix}$  cancel, and we are left with

$$(h_1 - g_1)\Omega.(B, S, W) = \sum_{\substack{S_1 \in \text{Orb}_{1' 2'}(\lambda \sqcup S) \\ S_1 \not\equiv \begin{smallmatrix} 2' & 2' \\ 1' & 1' \end{smallmatrix}}} \frac{1}{\kappa^2} z_{S, S_1} \left( \eta(q^{\Gamma(S_1[2'], S_1[1'])}) (B, S_1, W_1)_{(n, n-1)} \right. \\ \left. - \zeta(q^{\Gamma(S_1[2'], S_1[1'])}) (B, S_1, W_1)_{(1', 2')} \right),$$

where, in view of the structure of the  $1' 2'$ -orbit without the  $\begin{smallmatrix} 2' \\ 1' \end{smallmatrix}$  and  $\begin{smallmatrix} 2' & 1' \\ 1' \end{smallmatrix}$  configurations, the terms can be collected pairwise, and hence

$$(h_1 - g_1)\Omega.(B, S, W) = \sum_{\substack{S_1 \in \text{Orb}_{1'2'}(\lambda \sqcup S) \\ S_1 \not\equiv \begin{smallmatrix} 2' & 2' \\ 1' & 1' \end{smallmatrix}}} \frac{1}{\kappa^2} z_{S, S_1} y_{S, S_1}(B, S_1, W_1)_{(1', 2')}.$$

The factors that “compare” the  $z_{S, S_1}$  coefficients in front of similar terms are given by

$$y_{S, T} = -\zeta(q^{\Gamma(T[2'], T[1'])}) + \frac{\tilde{F}(q^{\Gamma(S[2'], T[2'])})}{\tilde{F}(q^{\Gamma(S[2'], T[1'])})} \frac{\tilde{\eta}(q^{\Gamma(S[2'], T[1'])})}{\tilde{\eta}(q^{\Gamma(S[2'], T[2'])})} \frac{\tilde{\zeta}(q^{\Gamma(S[2'], T[1'])})}{\tilde{\zeta}(q^{\Gamma(S[2'], T[2'])})} \eta(q^{-\Gamma(T[2'], T[1'])}),$$

where, to avoid proliferating cases in the formula for  $y_{S, T}$ , we define

$$\tilde{\eta}(x) = \begin{cases} 1, & x = 1, \\ \eta(x) & \text{otherwise,} \end{cases} \quad \tilde{\zeta}(x) = \begin{cases} 1, & x = 1, \\ \zeta(x) & \text{otherwise,} \end{cases} \quad \tilde{F}(x) = \begin{cases} -\frac{1}{\alpha\beta}, & x = 1, \\ F(x) & \text{otherwise.} \end{cases}$$

Using (B.7) and (5.3), we now conclude that  $(g_1 - h_1)\Omega.(B, S, W) = 0$  for all standard triples if and only if (5.4) holds. This completes the proof of **5.4**.

### 5.5.11. Remarks.

- (1) Defined for generic parameter values, the seminormal representation  $\mathcal{L}^{m, n, \lambda', \lambda}$  is irreducible, and is then readily identified with the irreducible  $\text{qw}\mathcal{B}_{m, n}$  representation defined by two Young diagrams  $(\lambda', \lambda)$ , of dimension  $\|\lambda'\| \|\lambda\| \binom{m}{f} \binom{n}{f} f!$ ,  $f = m - |\lambda'| = n - |\lambda|$  (where  $\|\lambda\|$  is the number of standard tableaux built on a Young diagram  $\lambda$ ). In particular, this gives the number of standard triples in  $\mathbf{T}_{m, n}(\lambda', \lambda)$ .
- (2) The matrices for all  $\text{qw}\mathcal{B}$  generators acquire denominators in the seminormal basis, which is the technical reason why seminormal representations cease to exist at certain parameter values. Studying these denominators (which are very explicit in our formulation) can be rather informative (cf., e.g., [45] in a different, but not unrelated context).

**5.6. Scalar product.** We next construct an invariant scalar product on each seminormal representation  $\mathcal{L}^{m, n, \lambda', \lambda}$ .

**5.6.1. Entropy.** We define the entropy of a pair  $(\mathbf{t}, w)$ , where  $\mathbf{t}$  is a Young tableau filled with  $1, \dots, n$  and  $w$  is an  $n$ -dimensional vector with nonzero components. We write  $\mathbf{t} = \sigma \mathbf{t}^s$ , where  $\mathbf{t}^s$  is the (row-reading) superstandard tableau and  $\sigma \in \mathbb{S}_n$  acts by permuting entries. We then define an  $\mathbb{S}_n$  action on  $w$  by specifying how the elementary transpositions act:

$$\sigma_{i, i+1}(w_1, \dots, w_n) = \frac{w_i}{w_{i+1}}(w_1, \dots, w_{i+1}, w_i, \dots, w_n)$$

(which is a representation of the symmetric group). Then the entropy  $E(\mathbf{t}, w)$  is the collection (unordered list) of all factors thus obtained in acting with  $\sigma$  on  $w$  (not the product of

factors). The inherent nonuniqueness does not affect the final result in view of the nature of the functions applied to the entropy in what follows.

If  $w$  is an  $n$ -dimensional vector, but a standard tableau  $t$  is filled not with  $1, \dots, n$  but with some numbers  $a_{i_1} < \dots < a_{i_n}$ , then there is a unique monotonic map  $\phi : (a_{i_1}, \dots, a_{i_n}) \rightarrow (1, \dots, n)$ , and we set  $E(t, w) = E(\phi t, w)$ . For example,  $\phi : \begin{array}{|c|c|c|} \hline 2 & 6 & 7 \\ \hline 4 & 9 & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array}$ .

If, further,  $t$  is an *antistandard* tableau (and, as in our case, is built on a skew shape), then we define  $\sigma$  by comparing with the super-antistandard row-reading tableau based on the same skew shape, and then apply  $\sigma$  to  $w$  by the same rule as above.

We apply this construction to standard triples as follows. Given a standard triple  $(B, S, W) \in \mathbf{T}_{m,n}(\lambda', \lambda)$  and the corresponding weight  $(\mu_1, \dots, \mu_{m+n-1}) = \text{Wt}(B, S, W)$ , we set

$$E(W) = E(W, (1, \mu_1, \dots, \mu_{n-1}))$$

(with 1 prepended to the first  $n - 1$  components of the weight so as to make an  $n$ -component vector).

Quite similarly,

$$E(B) = E(B, [\mu_n, \dots, \mu_{n+m-1}]_B),$$

where the vector comprises those components among  $\mu_n, \dots, \mu_{n+m-1}$  that correspond to the entries of  $B$  (not proportional to  $\theta$ ; see 5.2); the corresponding  $\phi$  map is understood here wherever needed.

Finally, to define the entropy of  $S$ , we extract  $\sigma$  from  $S = \sigma S^{\text{as}}$ , where  $S^{\text{as}}$  is a super-antistandard tableau as noted above. With this “antistandard option” indicated by a prime, we set

$$E(S) = E'(S, [\mu_{n+m-1}, \dots, \mu_n]_S),$$

where this time only the components corresponding to the entries of  $S$  (those proportional to  $\theta$ ) are selected and placed in reverse order, in accordance with their association with the entries of  $S$ .

**5.6.2. Examples.** For the second standard triple in 5.1.2, we have

$$W = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline 6 & & \\ \hline \end{array}, \quad w = (1, \mu_1, \dots, \mu_5) = (1, q, q^{-1}, q^{-2}, 1, q^2).$$

Then  $\sigma = \sigma_{3,4}\sigma_{2,3}$ , and we have

$$\begin{aligned} \sigma_{2,3}w &= q^2(1, q^{-1}, q, q^{-2}, 1, q^2), \\ \sigma_{3,4}(1, q^{-1}, q, q^{-2}, 1, q^2) &= q^3(1, q^{-1}, q^{-2}, q, 1, q^2), \end{aligned}$$

whence the entropy  $E(W) = \{q^3, q^2\}$ .

For the antistandard tableau entering the same standard triple, we have

$$S = \begin{array}{|c|c|} \hline & 4' \\ \hline 6' & 3' \\ \hline 1' & \\ \hline \end{array}, \quad w = (\mu_6, \mu_4, \mu_3, \mu_1) = (-\theta q^{-1}, -q^2 \theta, -\theta, -\theta q^{-2}).$$

The ordering in  $4', 6', 3', 1'$  differs from a super-antistandard by a single transposition, and hence the entropy is  $E(S) = \{\mu_6/\mu_4\} = \{q^{-3}\}$ .

As another example, we consider the standard triple

$$(B, S, W) = \left( \begin{array}{|c|c|} \hline 2' & 6' \\ \hline 5' & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 7' & 4' \\ \hline 3' & \\ \hline \end{array}, \begin{array}{|c|} \hline 1' \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & & \\ \hline 6 & & & \\ \hline \end{array} \right) \in \mathbf{T}_{7,7} \left( \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \right)$$

and the associated weight

$$\mu = (q, q^{-1}, q^{-2}, 1, q^2, q^{-3}; -\theta q^3, 1, -\theta q^{-2}, -\theta, q, q^{-1}, -\theta q^{-1}).$$

The corresponding entropies are

$$\begin{aligned} E(W) &= E(W, (1, q, q^{-1}, q^{-2}, 1, q^2, q^{-3})) = \{q^4, q^3, q^2, q^3, q^5\}, \\ E'(S) &= E\left(\begin{array}{|c|c|} \hline 4 & 3 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, (-\theta q^{-1}, -\theta, -\theta q^{-2}, -\theta q^3)\right) = \{q^{-4}, q^{-3}, q^{-5}\}, \\ E(B) &= E\left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, (1, q, q^{-1})\right) = \{q^2\}. \end{aligned}$$

**5.6.3. Mutual entropy.** Given a standard triple  $(B, S, W) \in \mathbf{T}_{m,n}(\lambda', \lambda)$ , we define the mutual entropy of  $B$  and  $S$  as the following set of  $q^k$ :

$$E(B, S) = \bigsqcup_{i' \in S} \bigsqcup_{\substack{j' \in B \\ j' < i'}} q^{\Gamma(S[i']) - \Gamma(B[j'])}.$$

This can be interpreted as follows. From the weight  $(\mu_1, \dots, \mu_{m+n-1})$ , we select the last  $m$  components  $\mu_n, \dots, \mu_{m+n-1}$ ; among these, there are  $f$  components  $\mu_{i_1}, \dots, \mu_{i_f}$  corresponding to the entries of  $S$ . For each such  $\mu_{i_r}$ , we take all ratios  $\mu_{i_r}/\mu'$ , where  $\mu'$  ranges over components *to the left of*  $\mu_{i_r}$  that correspond to the entries of  $B$ . The right-hand side of the last formula is the collection of all such ratios.

**5.6.4. Shape factor.** The scalar product that we define on standard triples in what follows is made of the entropies, which depend on the *tableaux*, and of a factor that depends only on the shape of the diagrams. We now define the shape factors for each  $\mathcal{L}^{m,n,\lambda',\lambda}$ .

For a standard triple  $(B, S, W) \in \mathbf{T}_{m,n}(\lambda', \lambda)$ , the shapes of  $S$  and  $W$  are related as  $s \equiv \bar{s} = \bar{w}/\lambda$ , and, as previously, we prefer dropping  $\bar{w}$  from the notation, replacing it with

$\lambda \sqcup s$ . The shapes in  $\mathbf{T}_{m,n}(\lambda', \lambda)$  are

$$(5.15) \quad \lambda \sqcup s = (\lambda_1 + f_1, \dots, \lambda_k + f_k; f_{k+1}, \dots, f_\ell),$$

where  $f_1 + \dots + f_\ell = f \equiv n - |\lambda|$ ,  $\ell \geq k$ , with  $f_1 \geq 0, \dots, f_k \geq 0$  and  $f_{k+1} \geq \dots \geq f_\ell \geq 1$  (we use a semicolon to separate rows  $i$  with  $\lambda_i > 0$ ).

We select a reference shape where  $s$  is a single row (of  $f$  boxes) that extends the top row of  $\lambda$ :

$$\lambda \sqcup s^{\text{ref}} = (\lambda_1 + f, \lambda_2, \dots, \lambda_k).$$

We set  $D(\lambda \sqcup s^{\text{ref}}) = 1$ , and define the shape factor  $D(\cdot)$  for all other shapes recursively.

For  $\lambda \sqcup s$  in (5.15), choose a corner  $\circ_J$  of  $\lambda \sqcup s$  in any of the rows  $2 \leq J \leq \ell$  such that  $f_J > 0$ , i.e., a corner not belonging to  $\lambda$ . (If there is no such corner, then we already have  $s = s^{\text{ref}}$ .) We let  $\Phi_J(\lambda \sqcup s)$  denote the diagram where the chosen corner is moved to the end of the top row; depending on where the corner was chosen,  $\Phi_J(\lambda \sqcup s)$  is one of the diagrams

$$(\lambda_1 + f_1 + 1, \lambda_2 + f_2, \dots, \lambda_J + f_J - 1, \dots, \lambda_k + f_k; f_{k+1}, \dots, f_\ell)$$

or

$$(\lambda_1 + f_1 + 1, \lambda_2 + f_2, \dots, \lambda_k + f_k; f_{k+1}, \dots, f_J - 1, \dots, f_\ell).$$

We then set

$$(5.16) \quad D(\lambda \sqcup s) = \frac{M_1(\Phi_J(\lambda \sqcup s))}{M_J(\lambda \sqcup s)} \frac{\tilde{M}_1(\Phi_J(\lambda \sqcup s), \lambda)}{\tilde{M}_J(\lambda \sqcup s, \lambda)} D(\Phi_J(\lambda \sqcup s)),$$

with

$$M_J(\Lambda) = \prod_{\square \in \Lambda(J+1, K)} H(q^{\Gamma(\square, \circ_J)}), \quad H(x) = -\frac{\eta(x)^2}{\alpha\beta F(x)},$$

$$\tilde{M}_J(\Lambda, \lambda) = \prod_{\substack{\square \in \Lambda(J+1, K) \\ \square \notin \lambda}} Z(q^{\Gamma(\square, \circ_J)}), \quad Z(x) = -\frac{\zeta(x)^2}{\alpha\beta F(x)}.$$

Here, for a Young diagram  $\Lambda$  and  $i \leq j$ , we let  $\Lambda(i, j) = \Lambda_i \sqcup \dots \sqcup \Lambda_j$  denote the part of the diagram made of the rows between (and including) the  $i$ th and  $j$ th rows. The products in the last two formulas are over boxes *strictly below the chosen corner* in the  $J$ th row: the boxes of  $\Lambda$  in the first case and the boxes of  $\Lambda/\lambda$  in the second.

We note that from (5.4), we have

$$(5.17) \quad H(x)Z(x)H(y)Z(y) = H(xy)Z(xy),$$

which is a consistency condition for the construction of  $D(\lambda \sqcup s)$ .

Formula (5.16) defines  $D(\cdot)$  for all shapes encountered in  $\mathbf{T}_{m,n}(\lambda', \lambda)$ , by gradually moving all boxes not belonging to  $\lambda$  to the top row.

**5.6.5. Example.** As an example of the calculation of the D factor, we take  $\lambda = (2, 2, 1)$  and choose a skew shape  $s$  such that  $\lambda \sqcup s = (4, 2, 2, 1)$ . Then

$$\begin{aligned} D\left(\begin{array}{c} \cdot \cdot \square\square \\ \cdot \cdot \square \\ \square \end{array}\right) &= H(q^4)H(q^5)^2H(q^6)Z(q^5)D\left(\begin{array}{c} \cdot \cdot \square\square\square \\ \cdot \square \end{array}\right) \\ &= H(q^4)H(q^5)^2H(q^6)Z(q^5) \cdot H(q^5)H(q^6)H(q^7)D\left(\begin{array}{c} \cdot \cdot \square\square\square\square \\ \cdot \end{array}\right) \\ &= H(q^4)H(q^5)^3H(q^6)^2H(q^7)Z(q^5). \end{aligned}$$

Moving the boxes around starting from another corner gives  $\frac{Z(q^7)H(q^4)H(q^5)^2H(q^6)^2H(q^7)^2}{Z(q^2)H(q^2)}$ , which is the same in view of (5.17).

**5.6.6. Theorem.** *The diagonal scalar product  $(\cdot, \cdot)$  on standard triples in  $\mathbf{T}_{m,n}(\lambda', \lambda)$  (the seminormal basis of  $\mathcal{L}^{m,n,\lambda',\lambda}$ ) defined by*

$$((B, S, W), (B, S, W)) = \frac{D(\lambda \sqcup \bar{S})}{\text{qdim}(\lambda \sqcup \bar{S}, q)} \prod_{\square \in S} \left(1 + \frac{\theta}{q^{\Gamma(\square)}}\right) \frac{\prod_{t \in E(W)} H(t) \prod_{v \in E(B)} Z(v)}{\prod_{u \in E'(S)} Z(u) \prod_{w \in E(B, S)} Z(w)}$$

is invariant under the action of  $\text{qwB}_{m,n}$  generators:  $(A \cdot t_1, t_2) = (t_1, A \cdot t_2)$  for any standard triples  $t_1$  and  $t_2$ , and  $A$  any of the  $\text{qwB}_{m,n}$  generators  $g_j$ ,  $\mathcal{E}$ , or  $h_i$ .

The quantum dimension  $\text{qdim}$  of a Young diagram is defined in **A.4**.

The proof is by direct verification. Showing the invariance under  $g_j$  and  $h_i$  amounts to a standard analysis of cases (which are somewhat more numerous for the  $g_j$ ). As regards the action of  $\mathcal{E}$ , we consider two skew tableaux  $S_1$  and  $S_2$  belonging to the orbit of the mobile element; they differ by the position of a single box. For the coefficients  $c_{S_i}^{(1)}$  in (5.7), their ratio  $c_{S_1}^{(1)}/c_{S_2}^{(1)}$  is evidently reproduced from the ratio of the products  $\prod_{\square \in S}$  in the formula for the scalar product. Moreover, for the coefficients  $c_{S_i}^{(2)}$ , we have

$$\frac{c_{S_1}^{(2)}}{c_{S_2}^{(2)}} = \frac{\text{qdim}(\lambda \sqcup \bar{S}_1, q)}{\text{qdim}(\lambda \sqcup \bar{S}_2, q)}, \quad q = -\frac{\beta}{\alpha},$$

leading to the desired result.

**5.6.7. Remark.** The formula for the scalar product considerably simplifies for the totally symmetric choice in (5.5): then  $H$ ,  $Z$ , and  $D$  are identically equal to 1, and

$$((B, S, W), (B, S, W)) = \frac{1}{\text{qdim}(\bar{W}, q)} \prod_{\square \in S} \left(1 + \frac{\theta}{q^{\Gamma(\square)}}\right)$$

(where, of course, the shape of the “white” tableau is  $\bar{W} = \lambda \sqcup \bar{S}$ ).

**5.6.8. Example.** For the 16-dimensional seminormal representation with  $m = 2$ ,  $n = 4$ , and  $(\lambda', \lambda) = (\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$ , we list some (a half) of its basis vectors—standard triples  $\mathbf{t}$ —and their scalar squares in the format  $\mathbf{t} \rightarrow (\mathbf{t}, \mathbf{t}) \rightarrow (\mathbf{t}, \mathbf{t})|_{\text{sym}}$ , where the last term is the form taken by the scalar product for the totally symmetric choice (5.5). For those  $\mathbf{t} = (B, S, W)$  where  $1' \in B$ , we have

$$\begin{aligned} \left( \begin{smallmatrix} \square \\ 1' \end{smallmatrix}, \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ 2 & 4 \end{smallmatrix} \right) &\rightarrow \frac{(\frac{\theta}{q} + 1)(q\theta + 1)(1 - q^2)^2(1 - q^3)\eta(q^2)^2\eta(q^3)^2}{(1 - q)q^3(1 - q^4)^2\alpha^2(\theta + 1)\zeta(-\theta)^2} \rightarrow \frac{(\theta + 1)(1 - q^2)}{q^2(1 - q^4)}, \\ \left( \begin{smallmatrix} \square \\ 1' \end{smallmatrix}, \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square \\ 2 & 3 & 4 \end{smallmatrix} \right) &\rightarrow \frac{(q\theta + 1)(\theta q^3 + 1)(1 - q^2)\eta(q^2)^2\eta(q^3)^2}{q^2(1 - q^4)\alpha^2(\theta q^2 + 1)\zeta(-q^2\theta)^2} \rightarrow \frac{(\theta q^2 + 1)(1 - q)}{q(1 - q^3)}, \\ \left( \begin{smallmatrix} \square \\ 1' \end{smallmatrix}, \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ 2 \\ 4 \end{smallmatrix} \right) &\rightarrow \frac{(\frac{\theta}{q^2} + 1)(\frac{\theta}{q} + 1)(1 - q^2)\eta(q^2)^2\eta(q^3)^2}{q^4(1 - q^4)\alpha^2(\frac{\theta}{q^2} + 1)\zeta(-\frac{\theta}{q^2})^2} \rightarrow \frac{(\frac{\theta}{q^2} + 1)(1 - q)}{q^3(1 - q^3)}, \\ \left( \begin{smallmatrix} \square \\ 1' \end{smallmatrix}, \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ 2 \\ 3 \end{smallmatrix} \right) &\rightarrow \frac{(\frac{\theta}{q^2} + 1)(\frac{\theta}{q} + 1)(1 - q^3)^2\eta(q^2)^2\eta(q^3)^4}{q^5(1 - q^4)^2\alpha^4(\frac{\theta}{q^2} + 1)\zeta(-\frac{\theta}{q^2})^2} \rightarrow \frac{(\frac{\theta}{q^2} + 1)(1 - q)}{q^3(1 - q^3)}, \end{aligned}$$

and for those with  $1' \in S$ ,

$$\begin{aligned} \left( \begin{smallmatrix} \square \\ 2' \end{smallmatrix}, \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ 1' & 4 \end{smallmatrix} \right) &\rightarrow \frac{(\theta + 1)(1 - q^2)^2(1 - q^3)\eta(q^2)^2\eta(q^3)^2}{(1 - q)q^4(1 - q^4)^2\alpha^4} \rightarrow \frac{(\theta + 1)(1 - q^2)}{q^2(1 - q^4)}, \\ \left( \begin{smallmatrix} \square \\ 2' \end{smallmatrix}, \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square \\ 1' & 2 & 3 \end{smallmatrix} \right) &\rightarrow \frac{(\theta q^2 + 1)(1 - q)}{q(1 - q^3)} \rightarrow \frac{(\theta q^2 + 1)(1 - q)}{q(1 - q^3)}, \\ \left( \begin{smallmatrix} \square \\ 2' \end{smallmatrix}, \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix}, \begin{smallmatrix} \square & \square & \square \\ 1' & 3 & 4 \end{smallmatrix} \right) &\rightarrow \frac{(\theta q^2 + 1)(1 - q^2)\eta(q^2)^2\eta(q^3)^2}{q^3(1 - q^4)\alpha^4} \rightarrow \frac{(\theta q^2 + 1)(1 - q)}{q(1 - q^3)}, \\ \left( \begin{smallmatrix} \square \\ 2' \end{smallmatrix}, \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix}, \begin{smallmatrix} \square & \square \\ 1' & 3 \end{smallmatrix} \right) &\rightarrow \frac{(\frac{\theta}{q^2} + 1)(1 - q^3)^2\eta(q^2)^2\eta(q^3)^4}{q^6(1 - q^4)^2\alpha^6} \rightarrow \frac{(\frac{\theta}{q^2} + 1)(1 - q)}{q^3(1 - q^3)}. \end{aligned}$$

**5.7. Theorem.** In a seminormal representation  $\mathcal{L}^{m,n,\lambda',\lambda}$ , the Jucys–Murphy elements  $J(n)_2, \dots, J(n)_{m+n}$  (see 2.4) act on the seminormal basis elements  $\mathbf{t} = (B, S, W)$  as

$$J(n)_j \cdot \mathbf{t} = W\mathbf{t}(\mathbf{t})_{j-1} \mathbf{t}, \quad j = 2, \dots, m + n,$$

where the weight of a standard triple is defined in 5.2.

## 5.8. Proof.

**5.8.1.** The first  $n - 1$  Jucys–Murphy elements  $J(n)_2, \dots, J(n)_n$  are Jucys–Murphy elements of the Hecke subalgebra  $\mathcal{H}_n \subset \text{qw}\mathcal{B}_{m,n}$ , and the assertion is well known [20]; we recall that it can be proved by induction on  $i$  in  $J(n)_i$ , based on definition (2.18), which can be equivalently rewritten as

$$\left( -\frac{1}{\alpha\beta} h_{n+1-i} + \frac{\alpha + \beta}{\alpha\beta} \right) J(n)_i = -\frac{1}{\alpha\beta} J(n)_{i-1} h_{n+1-i}, \quad 2 \leq i \leq n,$$

and the fact that when  $h_{n-i}$  acts nondiagonally, it gives rise to a new weight that differs from the original weight  $\mu$  by the transposition of two neighboring components,  $\mu_{i-1}$  and  $\mu_i$ .

For the remaining Jucys–Murphy elements  $J(n)_{n+1}, \dots, J(n)_{n+m}$ , the definition also implies the identities

$$\left(-\frac{1}{\alpha\beta}g_{j-1} + \frac{\alpha+\beta}{\alpha\beta}\right)J(n)_{n+j} = -\frac{1}{\alpha\beta}J(n)_{n+j-1}g_{j-1}, \quad 1 \leq j \leq m,$$

and it is also the case that whenever the action of  $g_{j'}$  produces a new standard triple  $((B, S)_{(j', j'+1)}, W)$ , its weight differs from the weight of  $(B, S, W)$  by the transposition of  $\mu_{n+j'-1}$  and  $\mu_{n+j'}$ . Hence, by the same argument, the statement of the theorem holds for  $J(n)_{n+2}, \dots, J(n)_{n+m}$  as soon as it holds for  $J(n)_{n+1}$ . It therefore remains to establish the claim for  $J(n)_{n+1}$ , i.e.,

$$(5.18) \quad J(n)_{n+1} \cdot \mathbf{t} = \text{Wt}(\mathbf{t})_n \mathbf{t}.$$

The relevant component of the weight is determined by the position of  $1'$  in the standard triple.

**5.8.2.** The Jucys–Murphy element  $J(n)_{n+1}$  is not related to the “lower” ones by a simple formula, and we instead use its explicit form found in 2.4.2:

$$J(n)_{n+1} = 1 - (\alpha + \beta)\kappa \sum_{s=1}^n (-\alpha\beta)^{s-1} h_{s-1,1}^{-1 \searrow} \mathcal{E} h_{1,s-1}^{-1 \swarrow}.$$

It readily follows that  $J(n)_{n+1}$  commutes with  $h_1, \dots, h_{n-1}$ . For  $h_2, \dots, h_{n-1}$ , which commute with  $\mathcal{E}$ , this is entirely a Hecke-algebra statement, and the commutativity for  $h_1$  is also immediate because, concentrating on the generators that do not commute with  $h_1$ , we have

$$\begin{aligned} h_1 \cdot h_2 h_1 \mathcal{E} h_1 h_2 &= h_2 h_1 h_2 \mathcal{E} h_1 h_2 \\ &= h_2 h_1 \mathcal{E} h_2 h_1 h_2 \\ &= h_2 h_1 \mathcal{E} h_1 h_2 \cdot h_1. \end{aligned}$$

Therefore,  $J(n)_{n+1}$  acts by an eigenvalue in each irreducible representation of  $\mathcal{H}_n$ . Because  $\mathcal{L}^{m,n,\lambda',\lambda}$  decomposes into a direct sum of irreducible  $\mathcal{H}_n$  representations, it remains to find these eigenvalues.

**5.8.3.** All calculations for  $J(n)_{n+1}$  can be done in  $\text{qw}\mathcal{B}_{1,n}$ , and, hence, in seminormal representations  $\mathcal{L}^{1,n,\emptyset,\lambda}$  (with  $|\lambda| = n-1$ ) and  $\mathcal{L}^{1,n,\square,\lambda}$  (with  $|\lambda| = n$ ). The second case is immediate, because  $\mathcal{E}$  then acts trivially, and therefore  $J(n)_{n+1}$  acts as identity; but for all standard triples  $(\boxed{1'}, S, W) \in \mathbf{T}_{1,n}(\square, \lambda)$ , the weight component in (5.18) is indeed 1.

We are therefore left with the first case, i.e., finding the eigenvalues of  $J(n)_{n+1}$  acting on standard triples

$$\mathbf{t} = (\emptyset, S, W),$$

where the skew shape  $S$  is a single box, attached to  $\lambda$  in one of the possible positions; the sought eigenvalue depends on that position (and the shape  $\lambda$ ).



By the invariance property of the scalar product, the sought eigenvalue is

$$(5.19) \quad \frac{(t, J(n)_{n+1} t)}{(t, t)} = 1 - (\alpha + \beta) \kappa \sum_{s=1}^n (-\alpha\beta)^{s-1} \frac{(h_{1,s-1}^{-1} t, \varepsilon h_{1,s-1}^{-1} t)}{(t, t)}.$$

For  $t$  of the above form, we choose  $W \sqcup n$  to be the row-reading superstandard tableau filled with  $1, \dots, n-1$ ; its shape, we recall, is  $\lambda$ . Because the single box of  $S$  carries  $1'$ , this standard triple has a mobile element. The nondiagonal part of the action of  $h_1^{-1}$  destroys the mobile element, and hence (as many times in the foregoing)  $h_1^{-1}$  effectively acts by an eigenvalue. Next, each  $h_i^{-1}$  in  $h_2^{-1} \dots h_s^{-1}$  acts by an eigenvalue whenever  $n-i$  and  $n-i+1$  are in the same row.

The left-hand side of (5.19) is therefore expressed as a sum over the rows of  $W$ . More precisely, let  $n$  stand in the  $K$ th row of  $W$ , and let also  $\square_i$  be the last box in the  $i$ th row (thus,  $\overline{W} = \lambda \sqcup \square_K$ ). The sum over  $i = 1, \dots, K-1$  in the next formula is over the rows above the  $K$ th one, plus the contribution of the part of the diagram below it. In addition to the notation  $\Lambda(i, j)$  introduced in 5.6.4, we let  $\Lambda(\geq j)$  and  $\Lambda(> j)$  denote the parts a Young diagram made of rows nonstrictly and strictly below a  $j$ th row. Then

$$\begin{aligned} \frac{(t, J(n)_{n+1} t)}{(t, t)} &= 1 - \left(1 + \frac{\theta}{q^{\Gamma(\square_K)}}\right) \frac{\prod_{\delta \in \lambda} (1 - q^{\Gamma(\delta, \square_K)})}{\prod_{\substack{\star \in \overline{\lambda} \\ \star \neq \square_K}} (1 - q^{\Gamma(\star, \square_K)})} \\ &\quad \times \left( \sum_{i=1}^{K-1} (1 - q^{-1}) q^{\Gamma(\square_i, \square_K)} \frac{\prod_{\substack{\star \in \overline{\lambda(i+1, K-1)}}} (1 - q^{\Gamma(\star, \square_K)})}{\prod_{\delta \in \lambda(i, K-1)} (1 - q^{\Gamma(\delta, \square_K)})} + \frac{\prod_{\substack{\star \in \overline{\lambda(>K)}}} (1 - q^{\Gamma(\star, \square_K)})}{\prod_{\delta \in \lambda(\geq K)} (1 - q^{\Gamma(\delta, \square_K)})} \right) \\ &= 1 - \left(1 + \frac{\theta}{q^{\Gamma(\square_K)}}\right) = -\frac{\theta}{q^{\Gamma(\square_K)}} = -\frac{\theta}{q^{\Gamma(S[1'])}}, \end{aligned}$$

which is the  $n$ th component of the weight defined in 5.2. This shows (5.18) and hence 5.7.

## 6. OUTLOOK

We have discussed the quantum walled Brauer algebras  $qwB_{m,n}$  starting with the endomorphism algebras of mixed tensor products. We constructed the link-state basis in  $qwB$  Specht modules, a Baxterization of the algebra (more precisely, of morphisms in an “ambient” category), and seminormal  $qwB$  representations for generic parameters of the algebra.

The results can be developed in various directions. Among these, we note finding a generalized seminormal basis for the special parameter values  $\theta = -\left(-\frac{\beta}{\alpha}\right)^r$ , and investigating lattice models/spin chains that can be constructed from the monodromy matrix obtained by Baxterization.

At  $\theta = -\left(-\frac{\beta}{\alpha}\right)^r$ ,  $r \in \mathbb{Z}$ , suitable quotients of the  $q\mathcal{WB}$  algebra centralize the action of  $q$ -deformed general linear Lie superalgebras on tensor products of their natural representations; the  $q\mathcal{WB}$  algebra becomes nonsemisimple, and the Jucys–Murphy elements acquire root vectors in Specht modules. A generalization of the seminormal basis can then be defined as a basis in which Jucys–Murphy elements take the standard Jordan form. The common Jordan structure of Jucys–Murphy elements then depends on  $r$  and is a subject to be investigated. A generalized seminormal basis is important, in particular, in finding the bimodule structure of the mixed tensor products of  $U_q\mathfrak{gl}(M|N)$  representations.

The “universal monodromy matrix” resulting from the proposed Baxterization relates to a “universal spin chain,” which yields specific, true spin chains (corresponding to the  $U_q\mathfrak{gl}(M|N)$  series) at special parameter values. It is of interest to develop an appropriate version of the Bethe-ansatz approach and to trace how the step-by-step degeneration descends from the universal model to a specific spin chain/lattice model with the chosen  $U_q\mathfrak{gl}(M|N)$  symmetry. Deeper insights are to be gained from the root-of-unity case (cf. [46]), where the centralizer of  $U_q\mathfrak{gl}(M|N)$  is expected to be a lattice  $W$ -algebra—a discretization (cf. [47]) of a  $W$ -algebra defined in two-dimensional conformal field theory in terms of the intersection of kernels of the screening operators corresponding to  $U_q\mathfrak{gl}(M|N)$ .

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## APPENDIX A. NOTATION AND CONVENTIONS

**A.1.** We let  $|\lambda|$  denote the number of boxes in a Young diagram or a skew shape (or in fact in a tableau built on any of these).

**A.2.** By the *position* of a box in a Young diagram or a tableau or a skew shape, we mean the coordinates of the box in a quadrant of  $\mathbb{Z}^2$ , assigned in accordance with the pattern

{1,1}	{1,2}	{1,3}	...
{2,1}	{2,2}		
{3,1}			...
⋮			

For a tableau  $\mathbf{t}$  containing a number  $k$ , we let  $\mathbf{t}[k]$  denote the position of  $k$  in  $\mathbf{t}$ . Clearly,  $\mathbf{t}[1] = \{1, 1\}$  for any nonempty standard tableau  $\mathbf{t}$ .

**A.3.** Given two positions  $\{i_1, j_1\}$  and  $\{i_2, j_2\}$ , we define their *hook distance* as

$$\Gamma(\{i_1, j_1\}, \{i_2, j_2\}) = i_1 - i_2 - j_1 + j_2.$$

For a single position  $(i, j)$ , we set

$$\Gamma(\{i, j\}) = i - j.$$

For a box  $\square$  in position  $\{i, j\}$ , we set  $\Gamma(\square) = \Gamma(\{i, j\})$ .

**A.4. Quantum dimension.** For a Young diagram  $\lambda = (\lambda_1, \dots, \lambda_k)$ , we define its quantum dimension as

$$\text{qdim}(\lambda, q) = \frac{\prod_{j=1}^k q^{(j-1)\lambda_j} \prod_{i=1}^{|\lambda|} (1 - q^i)}{\prod_{\square \in \lambda} (1 - q^{h(\square)})},$$

where  $h(\square)$  is the length of the hook drawn through a chosen box (and  $|\lambda| = \lambda_1 + \dots + \lambda_k$ ).

For example,

$$\text{qdim}\left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array}, q\right) = \frac{q^9(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)(1-q^7)(1-q^8)}{(1-q)^3(1-q^2)(1-q^3)(1-q^4)^2(1-q^6)}.$$

## APPENDIX B. HECKE ALGEBRAS

**B.1.** By the Hecke algebra  $\mathcal{H}_n = \mathcal{H}_n(\alpha, \beta)$ , we mean the Iwahori–Hecke algebra of type  $A_n$  over  $\mathbb{C}$ . It is the quotient of the braid group on  $n$  strands, with generators  $h_1, \dots, h_{n-1}$ , by the relations

$$(B.1) \quad (h_i - \alpha)(h_i - \beta) = 0, \quad 1 \leq i \leq n-1,$$

where  $\alpha$  and  $\beta$  are two complex numbers (typically, such that  $\alpha + \beta \neq 0$  and  $\alpha\beta \neq 0$ ). The algebra actually depends not on two but on one parameter, because  $\alpha$ ,  $\beta$ , and  $h_i$  can be rescaled simultaneously.

Thinking of the  $\mathcal{H}_n$  generators as coming from the braid group, we use the braid-group diagram notation for them:

$$h_1 = \begin{array}{c} \diagup \quad | \quad | \\ \diagdown \end{array} \cdots, \quad h_2 = \begin{array}{c} | \quad \diagup \quad | \\ \diagdown \end{array} \cdots, \quad \dots$$

Then Hecke relations (B.1) take the graphic form

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = -\alpha\beta \begin{array}{c} | \quad | \\ | \quad | \end{array} + (\alpha + \beta) \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array}.$$

**B.2. Jucys–Murphy elements.** In  $\mathcal{H}_n$ , we define a commuting family of Jucys–Murphy elements  $J_i$ ,  $1 \leq i \leq n$ , as follows:

$$(B.2) \quad J_1 = 1, \quad J_i = (-\alpha\beta)^{-1} h_{n+1-i} J_{i-1} h_{n+1-i},$$

$2 \leq i \leq n$ . In terms of braid diagrams,

$$J_2 = (-\alpha\beta)^{-2} \left[ \cdots \left| \begin{array}{c} \text{Diagram 1} \end{array} \right. \right], \quad J_3 = (-\alpha\beta)^{-2} \left[ \cdots \left| \begin{array}{c} \text{Diagram 2} \end{array} \right. \right], \quad J_4 = (-\alpha\beta)^{-3} \left[ \cdots \left| \begin{array}{c} \text{Diagram 3} \end{array} \right. \right],$$

and so on. Diagram manipulations immediately show that the  $J_i$  pairwise commute. The actual choice of the  $J_i$  family (which is not unique) and the labeling reflect our preferences in the main body of the paper.

**B.3. Specht modules of  $\mathcal{H}_n$ .** We essentially follow [48] in describing the action of  $\mathcal{H}_n$  on its Specht modules.

A Specht module  $S^\lambda$  of  $\mathcal{H}_n$  is associated with each Young diagram  $\lambda$ ,  $|\lambda| = n$ , and is defined for any values of  $\alpha$  and  $\beta$ . It has a basis labeled by all standard Young tableaux of shape  $\lambda$ . The  $\mathcal{H}_n$  generators  $h_k$  act on  $S^\lambda$  by first mapping into a larger space  $W^\lambda$  and then taking the quotient by a set of relations  $R$  such that  $W^\lambda/R = S^\lambda$ :

$$(B.3) \quad \begin{array}{ccc} S^\lambda & \xrightarrow{\quad} & W^\lambda \xrightarrow{\pi} S^\lambda, \\ & \searrow h_k & \nearrow \end{array}$$

where  $\pi$  is the canonical projection. The space  $W^\lambda$  is the linear span of all (not necessarily standard) Young tableaux obtained by filling  $\lambda$  with  $1, \dots, n$ , and  $R$  are the Garnir relations [49], which we describe below.

**B.3.1.** The first short arrow in (B.3) is defined as follows. We recall that every tableau  $t$  can be obtained by applying an element  $\sigma \in \mathbb{S}_n$  to a reference tableau  $t^s$  (which we choose as the row-reading superstandard tableau),  $t^\sigma = \sigma t^s$ , where  $\sigma$  acts just by permuting the entries. We write  $\sigma$  as a reduced (minimal-length) representation  $\sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k}$  in terms of elementary transpositions  $\sigma_i$ ,  $i = 1, \dots, n-1$ , and, accordingly,

$$t^\sigma = t^{i_1 i_2 \dots i_k} = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} t^s.$$

We call  $k$  the length of the tableau and write  $k = \ell(t^\sigma)$  (and set  $\ell(t^s) = 0$ ). Then the first arrow in (B.3) is

$$(B.4) \quad h_k t^\sigma = \begin{cases} \sigma_k t^\sigma, & \ell(\sigma_k t^\sigma) = \ell(t^\sigma) + 1, \\ (\alpha + \beta) t^\sigma - \alpha \beta \sigma_k t^\sigma & \text{otherwise.} \end{cases}$$

Clearly, this gives a nonstandard Young tableau in general. But modulo the Garnir relations  $R$ , any nonstandard Young tableau can be expressed as a linear combination of standard Young tableaux.

**B.3.2. Garnir relations.** The set of Garnir relations  $R$  consists of two subsets,  $R = R_\alpha \cup R_\beta$ .

The relations in  $R_\alpha$  are those that make the rows of  $\mathfrak{t}$  standard: for any row  $k_1, \dots, k_g$  with a “disorder”  $k_i > k_{i+1}$ , we order the offending numbers at the expense of the factor  $\alpha$  appearing in front of the tableau. Hence, if  $\mathfrak{t}$  is a tableau with the total of  $K$  instances of disorder in its rows, then the corresponding relation in  $R_\alpha$  is

$$\mathfrak{t} = \alpha^K \tilde{\mathfrak{t}},$$

where  $\tilde{\mathfrak{t}}$  is the corresponding row-standard tableau.

The relations in  $R_\beta$  allow linearly expressing any tableau with nonstandard columns as linear combinations of column-standard tableaux. For any tableau with a transposition in a column,

$$\mathfrak{t} = \begin{array}{ccccccc} \boxed{\cdot} & \dots & \dots & \dots & \dots & \dots & \dots & \boxed{\cdot} \\ & & & & & & & \vdots \\ & & & & & & & \boxed{\cdot} \\ & & & & & & & \vdots \\ & & & & & & & \boxed{\cdot} \\ & & & & & & & \vdots \\ & & & & & & & \boxed{\cdot} \\ & & & & & & & \vdots \\ & & & & & & & \boxed{\cdot} \\ & & & & & & & \vdots \\ & & & & & & & \boxed{\cdot} \end{array} \quad (b' = b - 1),$$

where  $x_1 > y_b$ . Let  $\mathbb{W}(x, y)$  be the set of all permutations of the form

$$(B.5) \quad \sigma = \begin{pmatrix} x_{j_1} & x_{j_2} & \dots & x_{j_r} \\ y_{i_1} & y_{i_2} & \dots & y_{i_r} \end{pmatrix}, \quad 1 \leq r \leq \min(a, b),$$

where  $(x_{j_1}, x_{j_2}, \dots, x_{j_r})$  and  $(y_{i_1}, y_{i_2}, \dots, y_{i_r})$  are ordered subsets of  $(x_1, \dots, x_a)$  and  $y = (y_1, \dots, y_b)$ . We set

$$L(\sigma) = r,$$

$$w(\sigma) = \sum_{k=1}^r \Gamma(x_{j_k}, y_{i_k}),$$

where the hook distance  $\Gamma(\cdot, \cdot)$  is defined in A.3. We then have the relation

$$\mathfrak{t} = - \sum_{\sigma \in \mathbb{W}(x, y)} \alpha^{L(\sigma)^2} \left( -\frac{\beta}{\alpha} \right)^{-w(\sigma)} \sigma \mathfrak{t},$$

where the  $\sigma$  act by permuting the entries. The set  $R_\beta$  contains all such relations.

This defines  $\pi$  and hence the action in (B.3): by the repeated use of Garnir relations, every tableau in the right-hand side of (B.4) is expressed as a linear combination of standard Young tableaux.

**B.3.3. Example.** We consider the tableau

$$t = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 7 \\ \hline 2 & 3 & 6 & \\ \hline \end{array}$$

with disorder in the second column. The corresponding permutations are then given by  $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ ,  $\begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix}$ ,  $\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$ ,  $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ ,  $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ ,  $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ ,  $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ , and  $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ , and applying the  $R_\beta$  relations yields

$$\begin{aligned} t = & -\frac{\beta^6}{\alpha^2} \begin{array}{|c|c|c|c|} \hline 1 & 4 & 2 & 3 \\ \hline 5 & 7 & 6 & \\ \hline \end{array} + \frac{\beta^5}{\alpha} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 3 \\ \hline 4 & 7 & 6 & \\ \hline \end{array} - \beta^4 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 & \\ \hline \end{array} - \frac{\beta^4}{\alpha^3} \begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 2 \\ \hline 7 & 3 & 6 & \\ \hline \end{array} \\ & + \frac{\beta^3}{\alpha^2} \begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 3 \\ \hline 2 & 7 & 6 & \\ \hline \end{array} + \frac{\beta^3}{\alpha^2} \begin{array}{|c|c|c|c|} \hline 1 & 4 & 2 & 7 \\ \hline 5 & 3 & 6 & \\ \hline \end{array} - \frac{\beta^2}{\alpha} \begin{array}{|c|c|c|c|} \hline 1 & 4 & 3 & 7 \\ \hline 2 & 5 & 6 & \\ \hline \end{array} - \frac{\beta^2}{\alpha} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 7 \\ \hline 4 & 3 & 6 & \\ \hline \end{array} + \beta \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 6 & \\ \hline \end{array}. \end{aligned}$$

After applying the  $R_\alpha$  relations to order the rows, we obtain

$$\begin{aligned} t = & -\alpha\beta^6 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline \end{array} + \alpha\beta^5 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 & \\ \hline \end{array} - \beta^4 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 & \\ \hline \end{array} - \alpha\beta^4 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & 7 & \\ \hline \end{array} \\ & + \alpha\beta^3 \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & 7 & \\ \hline \end{array} + \beta^3 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 & \\ \hline \end{array} - \beta^2 \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & 6 & \\ \hline \end{array} - \beta^2 \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 & 6 & \\ \hline \end{array} + \beta \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 6 & \\ \hline \end{array}. \end{aligned}$$

**B.4. Seminormal representations of  $\mathcal{H}_n(\alpha, \beta)$ .** We briefly review the facts that we need about the seminormal representations of Hecke algebras [30] (also see [27] and the references therein).

**B.4.1.** For a standard  $n$ -box tableau  $t$  filled with  $1, \dots, n$ , we define its weight

$$\text{wt}(t) = (\text{wt}_1(t), \dots, \text{wt}_{n-1}(t)) \in \mathbb{C}^{n-1}, \quad \text{wt}_i(t) = \left(-\frac{\beta}{\alpha}\right)^{\Gamma(i+1)}$$

For example,

$$t = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & 8 \\ \hline 5 & & \\ \hline 7 & & \\ \hline \end{array} \implies \text{wt}(t) = \left(-\frac{\beta}{\alpha}, -\frac{\alpha}{\beta}, 1, \frac{\beta^2}{\alpha^2}, \frac{\alpha^2}{\beta^2}, -\frac{\beta^3}{\alpha^3}, -\frac{\alpha}{\beta}\right)$$

**B.4.2.** For generic  $\alpha$  and  $\beta$ , given a Young diagram  $\lambda$ , the seminormal representation  $\mathcal{V}^\lambda$  is defined by specifying the  $\mathcal{H}_n(\alpha, \beta)$  action on a basis of standard tableaux of shape  $\lambda$ . The generators act on such a tableau  $t$  as<sup>15</sup>

$$(B.6) \quad h_{n-i} \cdot t = \begin{cases} \alpha t, & i \text{ and } i+1 \text{ are in the same row,} \\ \beta t, & i \text{ and } i+1 \text{ are in the same column,} \\ \frac{\alpha + \beta}{1 - \frac{w_{i-1}}{w_i}} t + \eta\left(\frac{w_{i-1}}{w_i}\right) t_{(i,i+1)} & \text{otherwise,} \end{cases}$$

<sup>15</sup>That the left-hand side involves  $h_{n-i}$  rather than the more standard  $h_i$  means that a Hecke algebra automorphism is applied to a standard construction, which is done for our purposes in the main text.

where  $w = \text{wt}(\mathbf{t})$ , and  $\mathbf{t} \mapsto \mathbf{t}_{(i,i+1)}$  is the transposition of  $i$  and  $i+1$ .<sup>16</sup> The function  $\eta(\cdot)$  is a convenient way to reflect the arbitrariness of rescaling the basis elements. For the Hecke-algebra property (B.1) to hold for (B.6), as is easy to see,  $\eta$  must satisfy the relation

$$(B.7) \quad \eta(x)\eta\left(\frac{1}{x}\right) = -\alpha\beta F(x),$$

where

$$(B.8) \quad F(x) = \frac{\left(1 + \frac{x\beta}{\alpha}\right)\left(1 + \frac{x\alpha}{\beta}\right)}{(1-x)^2}.$$

The same relation (B.7) suffices for the Yang–Baxter equation  $h_i h_{i+1} h_i = h_{i+1} h_i h_{i+1}$  to hold for the action in (B.6). To see this, note that  $\tilde{w} = \text{wt}(\mathbf{t}_{(i,i+1)})$  differs from  $w$  by the transposition of the  $(i-1)$ th and  $i$ th components. Verifying the Yang–Baxter equation then amounts to a straightforward analysis of the possible cases. For example, if both  $h_{n-i}$  and  $h_{n-i-1}$  act in accordance with the third line in (B.6), then

$$\begin{aligned} h_{n-i} h_{n-i-1} h_{n-i} \mathbf{t} - h_{n-i-1} h_{n-i} h_{n-i-1} \mathbf{t} &= (\alpha + \beta) \times \\ &\left( \frac{(\alpha + \beta)^2 w_i w_{i+1} (w_{i-1} w_{i+1} - w_i^2)}{(w_{i-1} - w_i)^2 (w_i - w_{i+1})^2} + \frac{w_{i+1} \left( \eta\left(\frac{w_i}{w_{i+1}}\right) \eta\left(\frac{w_{i+1}}{w_i}\right) - \eta\left(\frac{w_{i-1}}{w_i}\right) \eta\left(\frac{w_i}{w_{i-1}}\right) \right)}{w_{i-1} - w_{i+1}} \right) \mathbf{t}, \end{aligned}$$

which does vanish by virtue of (B.7). The other cases (where one of the generators acts by an eigenvalue) are verified similarly.

A symmetric choice for  $\eta(\cdot)$  satisfying (B.7) is

$$\eta(x) = \sqrt{-\alpha\beta F(x)}.$$

**B.4.3.** In the seminormal representation  $\mathcal{V}^\lambda$ , Jucys–Murphy elements (B.2) are diagonalized,

$$J_i \cdot \mathbf{t} = \text{wt}_{i-1}(\mathbf{t}) \mathbf{t}, \quad 2 \leq i \leq n,$$

for each standard tableau of the given shape  $\lambda$ .

## APPENDIX C. TWO-VARIATE IDENTITIES

We prove identities (5.8) and (5.9) by actually proving an apparently stronger statement in **C.1**, which implies that the identities in fact hold in a generalized, two-variate form, in which they border with generalized Pieri rules (see [50] and the references therein).

Instead of a single indeterminate  $q = -\beta/\alpha$  in Sec. 5, we introduce two variables  $x$  and  $t$  and associate them with vertical and horizontal distances  $\Delta_v$  and  $\Delta_h$  between boxes in a

<sup>16</sup>In the third line in (B.6),  $\mathbf{t}_{(i,i+1)}$  is again a standard tableau. The third line cannot occur for  $i = 1$  just because 1 and 2 are necessarily in the same row or column.

Young diagram. To facilitate the comparison with the Pieri rule formulas in the literature (see, e.g., [50]), we define the weight of a box in terms of its coleg and coarm:

$$\text{weight}(\square) = x^{l'(\square)} t^{a'(\square)}.$$

**C.1. Lemma.** *Let  $\lambda$  be a Young diagram with  $\ell$  corners  $\circ_i$ ,  $1 \leq i \leq \ell$ , and the addable boxes  $\star_k$ ,  $0 \leq k \leq \ell$ . Set  $\bar{y}_i = xt \text{weight}(\circ_i)$  and  $\bar{u}_k = \text{weight}(\star_k)$ . Then*

$$\sum_{k=0}^{\ell} \frac{1}{\bar{u}_k^j} \frac{1}{\prod_{\substack{i=0 \\ i \neq k}}^{\ell} (1 - \frac{\bar{u}_i}{\bar{u}_k})} = \begin{cases} 1, & j = 0, \\ 0, & 1 \leq j \leq \ell, \\ \frac{(-1)^\ell}{\bar{y}_1 \dots \bar{y}_\ell}, & j = \ell + 1. \end{cases}$$

To prove this, we simply note that for *any* pairwise distinct complex numbers  $u_0, u_1, \dots, u_\ell$ , the function

$$F_j(z) = \frac{z^j}{\prod_{i=0}^{\ell} (z - u_i)}$$

has the property that

$$\sum_{i=0}^{\ell} \text{res}_{u_i} F_j(z) = \sum_{i=0}^{\ell} \frac{u_i^j}{\prod_{\substack{k=0 \\ k \neq i}}^{\ell} (u_i - u_k)},$$

where the sum goes over all residues at finite  $z$ , but at the same time the residue at infinity is

$$\text{res}_{\infty} F_j(z) = \begin{cases} 0, & 1 \leq j \leq \ell - 1, \\ -1, & j = \ell, \\ -h_{j-\ell}(u_0, \dots, u_\ell), & j \geq \ell, \end{cases}$$

where  $h_n$  are the complete homogeneous symmetric polynomials

$$h_n(x) = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \dots x_{i_n}.$$

In applying this with  $u_i = \bar{u}_i$  as defined above, it remains to recall (see, e.g., [51]) that

$$\bar{u}_0 \dots \bar{u}_\ell = \bar{y}_1 \dots \bar{y}_\ell.$$

**C.2.** It follows from the lemma, in particular, that

$$(C.1) \quad \sum_{\star \in \overline{\Gamma\lambda}} \frac{\prod_{\substack{\circ \in \underline{\lambda} \\ \circ \in \star}} (1 - x^{\Delta_v(\star, \circ)} t^{\Delta_h(\star, \circ)})}{\prod_{\substack{\star' \neq \star}} (1 - x^{\Delta_v(\star, \star')} t^{\Delta_h(\star, \star')})} = 1$$

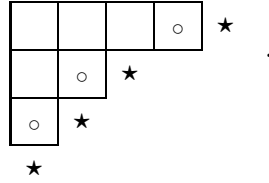


and

$$(C.2) \quad \sum_{\star \in \overline{\lambda}} \frac{\prod_{\substack{\circ \in \underline{\lambda}, \circ \neq \star}} (1 - x^{\Delta_v(\star, \circ)} t^{\Delta_h(\star, \circ)})}{\prod_{\star' \neq \star} (1 - x^{\Delta_v(\star, \star')} t^{\Delta_h(\star, \star')})} = 1,$$

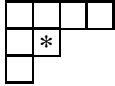
which are two-variate forms of (5.8) and (5.9). (As before,  $\underline{\lambda}$  are the corners of  $\lambda$  and  $\overline{\lambda}$  are the boxes addable to it, and  $\star$  is a fixed corner of  $\lambda$ ).

**C.3. Examples.** We illustrate (C.1) with  $\lambda = (4, 2, 1)$ , which has the following addable boxes  $\star$  and corners  $\circ$ :



Identity (C.1) then takes the form

$$(C.3) \quad \frac{(1-x)(1-x^2t^{-1})(1-x^3t^{-3})}{(1-xt^{-1})(1-x^2t^{-2})(1-x^3t^{-4})} + \frac{(1-t)(1-x)(1-x^2t^{-2})}{(1-x^{-1}t)(1-xt^{-1})(1-x^2t^{-3})} \\ + \frac{(1-t)(1-x^{-1}t^2)(1-xt^{-1})}{(1-x^{-2}t^2)(1-x^{-1}t)(1-xt^{-2})} + \frac{(1-t)(1-x^{-2}t^4)(1-x^{-1}t^3)}{(1-x^{-3}t^4)(1-x^{-2}t^3)(1-x^{-1}t^2)} = 1.$$

With a corner  $\star$  selected as  $(\lambda, \star) =$  , the corresponding identity (C.2) becomes the following “thinning” of (C.3):

$$\frac{(1-x)(1-x^3t^{-3})}{(1-xt^{-1})(1-x^2t^{-2})(1-x^3t^{-4})} + \frac{(1-t)(1-x^2t^{-2})}{(1-x^{-1}t)(1-xt^{-1})(1-x^2t^{-3})} \\ + \frac{(1-x^{-1}t^2)(1-xt^{-1})}{(1-x^{-2}t^2)(1-x^{-1}t)(1-xt^{-2})} + \frac{(1-t)(1-x^{-2}t^4)}{(1-x^{-3}t^4)(1-x^{-2}t^3)(1-x^{-1}t^2)} = 1.$$

Another, yet “thinner,” identity was needed in the proof of **5.5.3**.

**C.4. Remark.** Lemma **C.1** implies a (simple) part of the identities discussed in [50, 51] (also see the references therein). If we set

$$d_{\lambda^{(k)}, \lambda}(t, x) = \frac{1}{\bar{u}_k} \frac{\prod_{i=1}^{\ell} \left(1 - \frac{\bar{y}_i}{\bar{u}_k}\right)}{\prod_{\substack{i=0 \\ i \neq k}}^{\ell} \left(1 - \frac{\bar{u}_i}{\bar{u}_k}\right)},$$

then the identity

$$\sum_{k=0}^{\ell} d_{\lambda^{(k)}, \lambda}(t, x) = 1$$

follows from **C.1** by expanding the brackets in the numerator, after which we are left with only the top-degree term in the  $\bar{u}_i$ . Similarly, we also have

$$\sum_{k=0}^{\ell} \bar{u}_k d_{\lambda^{(k)}, \lambda}(t, x) = 1,$$

where just the constant term in the numerator contributes to produce the 1 in the right-hand side.

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LEBEDEV PHYSICS INSTITUTE

ASEMIKHA@GMAIL.COM, TIPUNIN@GMAIL.COM